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Manipushpak Mitra, Indrajit Ray and Souvik Roy

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Cardiff Business School
Cardiff University
Colum Drive
Cardiff CF10 3EU
United Kingdom
t: +44 (0)29 2087 4000
f: +44 (0)29 2087 4419
business.cardiff.ac.uk

A Characterisation of Trading Equilibria in Market Games*

Manipushpak Mitra†, Indrajit Ray‡§ and Souvik Roy¶

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Abstract

We provide a full characterisation of the set of *trading equilibria* (in which all goods are traded at a positive price) in a strategic market game (as introduced by Shapley and Shubik), for both the "buy and sell" and the "buy or sell" versions of this model under standard assumptions on the utility functions. We also interpret and illustrate our main equilibrium-characterising condition, using simple examples.

Keywords: strategic market game, trading equilibrium, buy and sell, buy or sell.

JEL Classification Numbers: C72, D44.

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†Economic Research Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700108, India. E-mail: mmitra@isical.ac.in; Fax: +91.33.2577.8893.

‡*Author for Correspondences.* Economics Section, Cardiff Business School, Cardiff University, Colum Drive, Cardiff CF10 3EU, UK. E-mail: rayi1@cardiff.ac.uk; Fax: +44.2920.874419.

§Department of Economics, Presidency University, 86/1 College Street, Kolkata, West Bengal 700073, India. E-mail: indrajit.cp@presiuniv.ac.in; Fax: +91.33.22410297.

¶Economic Research Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700108, India. E-mail: souvik.2004@gmail.com; Fax: +91.33.2577.8893.

1 INTRODUCTION

The strategic market game as introduced by Shapley and Shubik (Shubik,1973; Shapley, 1976; Shapley and Shubik 1977; Dubey and Shubik, 1977, 1978) perhaps is the most well-analysed model, outside the realm of the general equilibrium theory, to understand the formation of market-price as a strategic mechanism. In this non-cooperative game, individual traders (players) can influence the market price of any good through their buy and sell orders (strategies) when a specified commodity is used as the monetary medium ("money") for buying and selling all other commodities; the price of any good is simply the ratio of the amount of money and the amount of good at that trading-post. Other variations of this simple "buy and sell" game are "sell-all" (in which all commodities other than money are brought for sale) and "buy or sell" (in which traders cannot both buy and sell a particular good).

The strategic market game has the properties of a resource reallocation mechanism and the properties of a non-cooperative game. Traders select their (buying and selling) strategies to maximise their respective utilities; these strategies form an allocation (redistribution) of the existing resources. The definition of the non-cooperative equilibrium in this context is simply that of Nash equilibrium (in pure strategies) of the game, in which all traders, given the bids of others, are playing their respective best responses. As a consequence, "no-trade", by all players, always is an equilibrium for such a market game. This simple game has generated a vast literature over the last four decades and more complicated models and solution concepts out of this basic game have been analysed; as readers will appreciate, we are not listing all recent developments here (see survey papers by Giraud, 2003; Levando, 2012 and Dickson and Tonin, 2018).

Starting with Dubey and Shubik (1977), there is a literature on general existence and multiplicity of equilibria (see Peck, *et al*, 1992) for such games; moreover, the equilibria of such games are typically inefficient (see Dubey and Rogawski, 1992). Despite having a huge literature, we unfortunately lack a usable formal characterisation of the set of (Nash) equilibria even for a simple strategic market game. Perhaps as a result, we do not have many explanatory worked-out examples of equilibrium profile (with a possible exception of Ray, 2001 which is not really based on any formal characterisation either); even within the specific case of "bilateral oligopolies" (Gabszewicz and Michel, 1997), there is hardly any numerical example in the literature (see Dickson and Tonin, 2018 for examples derived from the first principle of Nash equilibrium behaviour in such models).

Given the importance of the strategic market games in different areas in microeconomics, macroeconomics and in finance, it would be natural to characterise the set of equilibrium outcomes for strategic market games. However, to the best of our knowledge, there is no such explicit study of equilibrium conditions (resulting from the utility maximisation problem of the traders). We address this issue in this paper by identifying conditions for trading equilibria and thereby fully characterising the set of trading equilibria for a basic buy and sell game (under some standard assumptions on the utility function of each trader). We also derive equilibrium conditions when traders are allowed to either buy or sell but are not allowed to do both in this game. Our key equilibrium condition applies to any interior profile in which no trader brings the entire endowment of any good (or money) to the market and thus everyone consumes at least some amount of all the goods and money.

The main result in this paper states that the equilibrium price of each good is a specific ratio, a constant for all traders. We interpret this characterising condition as follows: for any interior outcome to be an equilibrium requires that for each traded commodity j, the product of (i) the marginal rate of substitution between good j and money for any agent i and (ii) the ratio of the amount of good j bought and sold by all agents other than agent i, must be a constant across agents; moreover, this constant is indeed the (equilibrium) price of good j.

There are clear benefits of deriving such a characterisation for specific forms of market games, as we have demonstrated in this paper, using Cobb-Douglas utility functions. One can also easily construct specific numerical examples of equilibrium profiles for a given market game, as shown here. More importantly, one may check whether a particular outcome is an equilibrium outcome for a game or not. This will immediately help us to identify testability conditions (see the survey by Carvajal, *et al*, 2004) for such games, as analysed by Carvajal, *et al* (2013) for the Cournot model. In a parallel working paper (Mitra, *et al*, 2020), we address this particular issue.

2 MARKET GAMES

For the sake of completeness, we first briefly present our game, the strategic market game, *a la* Shapley and Shubik (Shubik (1973), Shapley (1976), and Shapley and Shubik (1977)).

A market is denoted by a four-tuple $\Xi = (N, X, E, U)$, where $N = \{1, \ldots, n\}$ is a finite set of traders; $X = (X_1 \times \ldots \times X_l) \times X_{l+1 \equiv m} \in \Re_+^{l+1}$ is the commodity space, where the $(l+1)^{\text{th}}$ commodity is the numeraire, "money"; $E = (e_i = (e_{i1}, \ldots, e_{il}, e_{im}) : i \in N)$ is an indexed collection of points in X representing the endowments of the traders; $U = (U_i : i \in N)$ is an indexed collection of functions from X to $\Re$ representing the utility functions of the traders.

Consider a market $\Xi = (N, X, E, U)$. Let us imagine l separate trading posts, one for each of the l commodities. Each individual i supplies q_{ij}, $q_{ij} \geq 0$, to each trading post $j \in \{1, \ldots, l\}$. Let $Q(j) = \sum_{i \in N} q_{ij}$, assumed to be positive, for all $j \in \{1, \ldots, l\}$. Denote $q_i := (q_{i1}, \ldots, q_{il})$. Each trader $i \in N$ makes bids by allocating amounts b_{ij} of his money (that is, the $(l+1)$-th commodity) to trading post j, for each $j \in \{1, \ldots, l\}$. We shall denote his buying strategy by the vector $b_i = (b_{i1}, \ldots, b_{il})$, with the constraints (a) $\sum_{j=1}^{l} b_{ij} \leq e_{im}$ and (b) $b_{ij} \geq 0$. The price emerges as a result of the simultaneous bids of all buyers, specifically $p_j = (B(j)/Q(j))$ where $B(j) := \sum_{i \in N} b_{ij}$.

With slight abuse of notation, define $(q, b) := ((q_i, b_i)_{i\in N})$ as an indexed collection of strategies or a strategy profile.

Given a market $\Xi = (N, X, E, U)$, if we assume that each trader can either buy or sell but not both, then we have a *buy or sell strategic market game*. The difference between a strategic market game and a buy or sell strategic market game is just in terms of admissible strategy profiles; while any strategy profile (q, b) for a buy or sell strategic market game is also a strategy profile for the strategic market game, the converse is not true.

Given a collection of strategies $((q_k, b_k)_{k\in N\setminus\{i\}})$ of all agents other than i, denote:

$x_i(q, b) = (x_{i1}(q, b), \dots, x_{il}(q, b), x_{im}(q, b)) \in X$, where,

$x_{ij}(q, b) = e_{ij} - q_{ij} + (b_{ij}/p_j)$ for each $j \in \{1, \dots, l\}$, and $x_{im}(q, b) = e_{im} - \sum_{j=1}^{l} b_{ij} + \sum_{j=1}^{l} p_j q_{ij}$.

Now agent i's utility maximisation problem (UMP) is to choose (q_i, b_i) to maximise $U_i(x_i(q, b))$ subject to $q_{ij} \in [0, e_{ij}]$, $b_{ij} \geq 0$ for each $j = 1, \dots, l$, and $\sum_{j=1}^{l} b_{ij} \leq e_{im}$.

Definition 1 *Given a market $\Xi = (N, X, E, U)$, a strategy profile (b, q) is a trading equilibrium of the corresponding market game if the following two conditions hold:*

1. *For each good $j \in \{1, \dots, l\}$, $p_j = \left(\sum_{i\in N} b_{ij} / \sum_{i\in N} q_{ij}\right) > 0$.*
2. *For each agent $i \in N$, given $((q_k, b_k)_{k\in N\setminus\{i\}})$ of all agents in $N\setminus\{i\}$, (q_i, b_i) is a solution to agent i's UMP.*

The strategy profile (q, b) with $q_{ij} = b_{ij} = 0$, for all $i \in N$ and all $j \in \{1, \dots, l\}$ is trivially a Nash equilibrium. Here all traders are inactive in the sense that they are neither buying nor selling any commodities.

We now make a couple of assumptions on the utility function, $U_i(x_i(q, b))$, for each trader $i \in N$, for the rest of the paper.

Assumption 1a. $U_i(x_i(q, b))$, for any $i \in N$, is continuously differentiable.

Consider any $i \in N$ and fix any trading decision $(q_j, b_j)_{j\in N\setminus\{i\}} \in \Re_{++}^{2l\times(n-1)}$ for all agents other than i. For each $(q_i, b_i) \in \Re_{++}^{2l}$, define $V_i(q_i, b_i) := U_i(x_i((q_i, b_i), (q_j, b_j)_{j\in N\setminus\{i\}}))$. Therefore, for any given trading decision of others, $(q_j, b_j)_{j\in N\setminus\{i\}} \in \Re_{++}^{2l\times(n-1)}$, the associated function $V_i : \Re_{++}^{2l} \to \Re$ represents the utility that trader i can have for different choices of (q_i, b_i).

The gradient vector of the function $V_i(q_i, b_i)$ is denoted by:

$\nabla V_i(q_i, b_i) = \left(\frac{\partial V_i(q_i,b_i)}{\partial q_{i1}}, \dots, \frac{\partial V_i(q_i,b_i)}{\partial q_{il}}, \frac{\partial V_i(q_i,b_i)}{\partial b_{i1}}, \dots, \frac{\partial V_i(q_i,b_i)}{\partial b_{il}}\right)$.

Observe that for any $(q_i, b_i), (q_i', b_i') \in \Re_{++}^{2l}$, $\nabla V_i(q_i, b_i)\cdot[(q_i', b_i') - (q_i, b_i)] = \sum_{k=1}^{l}(q_{ik}' - q_{ik})\frac{\partial V_i(q_i,b_i)}{\partial q_{ik}} + \sum_{k=1}^{l}(b_{ik}' - b_{ik})\frac{\partial V_i(q_i,b_i)}{\partial b_{ik}}$.

Definition 2 *A continuously differentiable utility function $U_i(.)$ is pseudo-concave if for any given $(q_j, b_j)_{j\in N\setminus\{i\}} \in \Re^{2l\times(n-1)}_{++}$, the associated real-valued differentiable function $V_i : \Re^{2l} \rightarrow \Re$ satisfies the following property:*

for all $(q_i, b_i), (q_i', b_i') \in \Re^{2l}_{++}$ such that $\sum_{k=1}^{l}(q_{ik}' - q_{ik})\frac{\partial V_i(q_i,b_i)}{\partial q_{ik}} + \sum_{\mathrm{k=1}}^{\mathrm{l}}(\mathrm{b}_{\mathrm{ik}}' - \mathrm{b}_{\mathrm{ik}})\frac{\partial \mathrm{V_i(q_i,b_i)}}{\partial \mathrm{b_{ik}}} \leq 0$, we have $V_i(q_i', b_i') \leq V_i(q_i, b_i)$.

Pseudo-concavity implies a restriction on the function $V_i(.,.)$. As is well-known, note that, if the utility function $U_i(x_i)$ is twice differentiable in (q_i, b_i), then the property of pseudo-concavity is equivalent to that of quasi-concavity (see Chapter 2, page 88 in Kall and Mayer, 2006).

Assumption 1b. $U_i(.)$, for any $i \in N$, is pseudo-concave.

The assumption of pseudo-concavity of the utility function of any trader i in this game ensures that the Kuhn-Tucker conditions are sufficient for the trader i's utility maximisation problem and thus will be used to prove our results in the next section. Specifically, if the objective function is pseudo-concave and all constraints are quasi-convex, then Kuhn-Tucker necessary conditions are also sufficient for a maximum. One may not in general know whether the objective function in a maximisation problem is pseudo-concave or not (see Chapter 9, pages $559 - 560$ in Miller, 2011). In such a case, one has to identify the set of solutions to the optimisation problem by trying out all the solutions generated by the Kuhn-Tucker necessary conditions.

For any strategy profile (q, b), for any $i \in N$, call $Q_{-i}(j) = Q(j) - q_{ij}$ and $B_{-i}(j) = B(j) - b_{ij}$. Following our on-going (slightly abusing) notations, we may write $(q, b) := ((q_i, Q_{-i}), (b_i, B_{-i}))$. We now identify the following crucial ratio for any $i \in N$ and any $j = 1, \ldots, l$:

$\Delta_{ij}(q, b) := \Delta_{ij}((q_i, Q_{-i}), (b_i, B_{-i})) = [B_{-i}(j)\,\{\partial U_i(x_i(q, b))/\partial x_{ij}\}]\,/\,[Q_{-i}(j)\,\{\partial U_i(x_i(q, b))/\partial x_{im}\}]$.

Finally, we identify a class of strategy profiles for a market game that we call *interior* for which the final allocations for all traders are interior points.

Definition 3 *A strategy profile (q, b) is a $\mathcal{B}$-profile if there exists $i \in N$ and there exists $j \in \{1, \ldots, l\}$ such that $q_{ij} = 0$ and $\sum_{k=1}^{l} b_{ik} = e_{im}$.*

In a $\mathcal{B}$-profile, there must be at least one trader who spends the entire money endowment and does not sell at least one good (and thus consumes at least as much the endowment for that good).

Definition 4 *A strategy profile (q, b) is a $\mathcal{Q}$-profile if there exists $i \in N$ and there exists $j \in \{1, \ldots, l\}$ such that $b_{ij} = 0$ and $q_{ij} = e_{ij}$.*

In a $\mathcal{Q}$-profile, there must be at least one trader who sells the whole endowment for at least one good and does not spend any money to buy back that good (and hence does not consume that good in the final allocation).

Definition 5 *A strategy profile* (q, b) *is said to be an interior strategy profile if it is neither a* $\mathcal{B}$*-profile nor a* $\mathcal{Q}$*-profile.*

In an interior profile, no trader brings the entire endowment of any good (or money) to the market and thus everyone consumes at least some amount of all the goods and money. For the case of a market game with two-goods (that is, one commodity and money, when $l = 1$), an interior strategy profile is given by $((q_i, b_i)_{i \in N})$ such that $0 < q_i < e_{i1}$ and $0 < b_i < e_{im}$, for all $i \in N$.

For any strategy profile (q, b) and any $j \in \{1, \ldots, l\}$, let $\mathcal{Q}_j(q, b) = \{i \in N \mid b_{ij} = 0 \; \& \; q_{ij} = e_{ij}\}$ and $\mathcal{B}_j(q, b) = \{i \in N \mid q_{ij} = 0 \; \& \; \sum_{k=1}^{l} b_{ik} = e_{im}\}$. By Definition 5, for any interior strategy profile (q, b), $\mathcal{Q}_j(q, b) \cup \mathcal{B}_j(q, b) = \emptyset$ for all $j \in \{1, \ldots, l\}$, whereas, for any non-interior strategy profile (q, b), there exists $j \in \{1, \ldots, l\}$ such that $\mathcal{Q}_j(q, b) \cup \mathcal{B}_j(q, b) \neq \emptyset$.

3 RESULTS

For the sake of clarity, we state and prove our main result in several steps. We first state a couple of lemmata about bids (b) and offers (q) at a trading equilibrium as below. The first lemma is related to the offers (q) at equilibrium.

Lemma 1 *At a trading equilibrium, for any* $i \in N$ *and any* $j \in \{1, \ldots, l\}$,

(a) *if* $q_{ij} \in (0, e_{ij})$, *then* $p_j^2 = \Delta_{ij}(q, b)$,

(b) *if* $q_{ij} = 0$, *then* $p_j^2 \leq \Delta_{ij}(q, b)$,

(c) *if* $q_{ij} = e_{ij} > 0$, *then* $p_j^2 \geq \Delta_{ij}(q, b)$.

Similarly, we provide conditions related to the bids (b) at equilibrium in our next lemma.

Lemma 2 *At a trading equilibrium, for any* $i \in N$ *and any* $j \in \{1, \ldots, l\}$,

(a) *if* $\sum_{k=1}^{l} b_{ik} < e_{im}$, *then* $p_j^2 \geq \Delta_{ij}(q, b)$,

(b) *if* $b_{ij} > 0$, *then* $p_j^2 \leq \Delta_{ij}(q, b)$.

The proofs of Lemma 1 and Lemma 2 heavily use the Kuhn-Tucker conditions for the optimisation problem for each individual trader. These proofs have been postponed to the Appendix of this paper. Above lemmata lead to the following proposition.

Proposition 1 *A strategy profile* (q, b) *is a trading equilibrium if and only if the following conditions hold for any* $i \in N$ *and any* $j \in \{1, \ldots, l\}$*:*

(K1) If $q_{ij} \in (0, e_{ij})$, then $p_j^2 = \Delta_{ij}(q, b)$.

(K2) If $q_{ij} = 0$ and $\sum_{k=1}^{l} b_{ik} < e_{im}$, then $p_j^2 = \Delta_{ij}(q, b)$.

(K3) If $q_{ij} = 0$ and $\sum_{k=1}^{l} b_{ik} = e_{im}$, then $p_j^2 \leq \Delta_{ij}(q, b)$.

(K4) If $q_{ij} = e_{ij}$ and $b_{ij} > 0$, then $p_j^2 = \Delta_{ij}(q, b)$.

(K5) If $q_{ij} = e_{ij}$ and $b_{ij} = 0$, then $p_j^2 \geq \Delta_{ij}(q, b)$.

Proposition 1 asserts that given our Assumptions 1a and 1b and given that the constraints in the individual maximisation problem are linear (hence, quasi-convex), the Kuhn-Tucker conditions are both necessary and sufficient to characterise equilibrium outcomes. The necessary conditions stated in Proposition 1, ($K1$)-($K5$), follow immediately from Lemma 1 and Lemma 2. In particular, from Lemma 1(a), we get ($K1$); from Lemma 1(b) and Lemma 2(a), we get condition ($K2$). Given condition ($K2$), from Lemma 2(b), we get ($K3$). From Lemma 1(c) and Lemma 2(b), we get condition ($K4$). Given condition ($K4$), from Lemma 1(c), we get ($K5$). The complete proof of Proposition 1 would thus require confirmation of the sufficiency part (that ($K1$)-($K5$) are sufficient for maximisation), using pseudo-concavity of the utility functions. The details of this part of the proof is in the Appendix.

We are now ready to state our main result.

Theorem 1 *Consider a market game with a market $\Xi = (N, X, E, U)$, under Assumptions* 1a *and* 1b.

1. *An interior strategy profile (q, b) is a trading equilibrium for this game if and only if $p_j^2 = \Delta_{ij}(q, b)$, for any $i \in N$ and any $j \in \{1, \dots, l\}$.*

2. *A non-interior strategy profile (q, b) is a trading equilibrium for this game if and only if we have the following for all $i \in N$:*

   (a) $p_j^2 \leq \Delta_{ij}(q, b)$, for any j for which $i \in \mathcal{B}_j(q, b)$,

   (b) $p_j^2 \geq \Delta_{ij}(q, b)$, for any j for which $i \in \mathcal{Q}_j(q, b)$ and

   (c) $p_j^2 = \Delta_{ij}(q, b)$, for any j for which $i \notin [\mathcal{Q}_j(q, b) \cup \mathcal{B}_j(q, b)]$.

Using Proposition 1, the proof of Theorem 1 is now immediate. We have mentioned the specific details in the Appendix.

3.1 Buy or Sell

We could easily rephrase our main theorem for any buy or sell market game as well.

For any given market $\Xi = (N, X, E, U)$, a strategy profile (q, b) for a strategic game is also feasible for a buy or sell strategic market game if and only if $i \in N$ and all $j \in \{1, \ldots, l\}$, $q_{ij} \geq 0$, $b_{ij} \geq 0$ and $q_{ij}.b_{ij} = 0$.

Subject to the above restriction, we can define $\mathcal{B}$-profile, $\mathcal{Q}$-profile and the interior strategy profile for any buy or sell market game exactly the same way as we did for a strategic market game earlier. To identify the conditions for a trading equilibrium for such a buy or sell model, we have the following result.

Theorem 2 *Consider a buy or sell strategic market game with a market* $\Xi = (N, X, E, U)$*, under Assumptions* 1a *and* 1b*.*

1. *An interior strategy profile* (q, b) *for this game is a trading equilibrium if and only if* $p_j^2 = \Delta_{ij}(q, b)$*, for any* $i \in N$ *and any* $j \in \{1, \ldots, l\}$*.*
2. *A non-interior strategy profile* (q, b) *for this game is a trading equilibrium if and only if we have the following for all* $i \in N$*:*
   (a) $p_j^2 \leq \Delta_{ij}(q, b)$*, for any* j *for which* $i \in \mathcal{B}_j(q, b)$ *and*
   (b) $p_j^2 \geq \Delta_{ij}(q, b)$*, for any* j *for which* $i \in \mathcal{Q}_j(q, b)$*.*

Proof of Theorem 2 is very similar to that Theorem 1 (using Proposition 1) above and thus has been postponed to the Appendix.

3.2 Interpretation

We now interpret and illustrate our key equilibrium condition, $\Delta_{ij}(q, b) = p_j^2$, obtained for a trading equilibrium with interior strategy profile (the first condition in Theorem 1).

A simplification of this condition is:

$$\frac{\partial U_i(x(q,b))}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial q_{ij}} = -\frac{\partial U_i(x(q,b))}{\partial x_{im}} \frac{\partial x_{im}}{\partial q_{ij}}. \tag{1}$$

Condition (1) states that, in equilibrium, the marginal rise in utility due to a rise in the consumption of x_{ij} caused by an incremental fall in q_{ij} must be equal to the absolute value of the marginal fall in utility due to a fall in the consumption of x_{im} caused by this incremental fall in q_{ij}. Using $p_j \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial x_{ij}}{\partial q_{ij}} = 0$ and $p_j \frac{\partial x_{im}}{\partial b_{ij}} + \frac{\partial x_{im}}{\partial q_{ij}} = 0$, one can rewrite condition (1) as follows:

$$\frac{\partial U_i(x(q,b))}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial b_{ij}} = -\frac{\partial U_i(x(q,b))}{\partial x_{im}} \frac{\partial x_{im}}{\partial b_{ij}}. \tag{2}$$

Like condition (1), condition (2) has a similar interpretation in terms of b_{ij}. The general requirements that $p_j \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial x_{ij}}{\partial q_{ij}} = 0$ and $p_j \frac{\partial x_{im}}{\partial b_{ij}} + \frac{\partial x_{im}}{\partial q_{ij}} = 0$ ensures that we have only one equilibrium condition for interior strategy profiles, that is, $\Delta_{ij} = p_j^2$.

Consider now (a) of the second condition in Theorem 1 obtained for a $\mathcal{B}$-profile, which states that, if, in equilibrium, we have $q_{ij} = 0$ and $\sum_{k=1}^{l} b_{ik} = e_{im}$, then either $p_j^2 = \Delta_{ij}(q, b)$ or $p_j^2 < \Delta_{ij}(q, b)$. In case, in equilibrium, we end-up with $p_j^2 < \Delta_{ij}(q, b)$, then it is not possible to equate p_j^2 and $\Delta_{ij}(q, b)$; neither by decreasing q_{ij} (since, $q_{ij} = 0$), nor by increasing b_{ij} (since, $\sum_{k=1}^{l} b_{ik} = e_{im}$).

Similarly, consider (b) of the second condition in Theorem 1 obtained for a $\mathcal{Q}$-profile, which states that, if, in equilibrium, we have $q_{ij} = e_{ij}$ and $b_{ij} = 0$, then either $p_j^2 = \Delta_{ij}(q, b)$ or $p_j^2 > \Delta_{ij}(q, b)$. Here as well, in case, in equilibrium, we end-up with $p_j^2 > \Delta_{ij}(q, b)$, then it is not possible to equate p_j^2 and $\Delta_{ij}(q, b)$; neither by increasing q_{ij} (since, $q_{ij} = e_{ij}$) nor by decreasing b_{ij} (since, $b_{ij} = 0$).

The equilibrium conditions for any interior condition are now quite transparent. It requires that for each traded commodity j, the marginal rate of substitution (MRS) between good j and money for any agent i ($MRS_{x_{ij}, x_{im}}$) times the ratio of the amount of good j bought and sold by all agents other than i must be a constant across agents. Moreover, this constant is the equilibrium price of good j.

Formally, for each commodity $j \in \{1, \ldots, l\}$,

$$p_j^* = \frac{\left(\frac{B_{-i}(j)}{p_j^*}\right) \frac{\partial U_i}{\partial x_{ij}}}{Q_{-i}(j) \frac{\partial U_i}{\partial x_{im}}} = \left(\frac{B_{-i}(j)}{p_j^* Q_{-i}(j)}\right) MRS_{x_{ij}, x_{im}}(x^*(b^*, q^*)) \; \forall \; i \in N.$$

3.3 Illustration (C-D utility)

Consider a specific market, denoted by Ξ_2^{C-D}, with just two agents, 1 and 2, with the following preferences and endowments: for agent 1, $U_1(x, y) = A_1 x^{\alpha_1} y^{1-\alpha_1}$, $A_1 > 0$, $\alpha_1 \in (0, 1)$, $\omega^1 = (\omega_x^1, \omega_y^1) >> 0$ and for agent 2, $U_2(x, y) = A_2 x^{\alpha_2} y^{1-\alpha_2}$, $A_2 > 0$, $\alpha_2 \in (0, 1)$, $\omega^2 = (\omega_x^2, \omega_y^2) >> 0$. We have the following characterisation for any interior equilibrium for the corresponding market game.

Proposition 2 *Any strategy profile (q, b) with $b_1 - q_1 = q_2 - b_2 = k > 0$ (implying $b_1 > k$ and $q_2 > k$) for a market game with the market Ξ_2^{C-D} constitutes an interior equilibrium with the equilibrium price $p = (b_1 + b_2)/(q_1 + q_2) = 1$ if and only if:*

$$\alpha_1 b_2 \omega_y^1 = (1 - \alpha_1) q_2 \omega_x^1 + k[\alpha_1 b_2 + (1 - \alpha_1) q_2], \; \omega_x^1 \geq q_1, \; \omega_y^1 \geq b_1 \geq k. \tag{3}$$

$$(1 - \alpha_2) q_1 \omega_x^2 = \alpha_2 b_1 \omega_y^2 + +k[\alpha_2 b_1 + (1 - \alpha_2) q_1], \; \omega_x^2 \geq q_2 \geq k, \; \omega_y^2 \geq b_2. \tag{4}$$

The proof is in the Appendix.

3.4 Examples

We first present a numerical example using the conditions in Proposition 2.

Example 1 *Consider a market game with two agents 1 and 2 with the following preferences and endowments: for agent 1, $U_1(x, y) = A_1 x^{\frac{1}{3}} y^{\frac{2}{3}}$, $A_1 > 0$, $\omega_x^1 = 2t_1$ and $\omega_y^1 = 5t_1 + 7$ with $t_1 \geq 1$ and for* *agent* 2*,* $U_2(x,y) = A_2 x^{\frac{3}{4}} y^{\frac{1}{4}}$*,* $A_2 > 0$*,* $\omega_x^2 = 48t_2 + 14$ *and* $\omega_y^2 = 8t_2$ *with* $t_2 \geq 1$*. Then the trade vector* $(b;q) = (b_1 = 4, q_1 = 2; b_2 = 8, q_2 = 10)$ *is an equilibrium strategy profile with equilibrium price* $p^* = 1$*.*

One may check the conditions in Proposition 2 to show that the strategy profile in Example 1 $(b_1 = 4, q_1 = 2; b_2 = 8, q_2 = 10)$ with $k = 2$ and $p = 1$ is indeed an equilibrium. Note that this profile is in equilibrium for a range of games with different endowments, determined by the choice of parameters, $t_1 \geq 1$ and $t_2 \geq 1$; in particular, for $t_1 = t_2 = 1$, we have the endowment vectors $(2, 12)$ and $(62, 8)$.

Our final example is from Dickson and Tonin (2018) and it uses a "bilateral oligopoly" (Gabszewicz and Michel, 1997) in which all agents have positive endowments in only one of the two goods.

Example 2 *Consider a market game with two goods and four agents of who agents* 1 *and* 2 *are identical with utility function and endowment given by* $\ln(1+x) + y$*,* $(3,0)$ *and agents* 3 *and* 4 *are identical with preferences and endowments* $3x - \frac{1}{2}x^2 + y$*,* $(0,5)$*. The strategy profile given by the trade vector* $(q_1 = q_2 = \frac{7-\sqrt{17}}{2}; b_3 = b_4 = \sqrt{17} - 3)$ *is an equilibrium price* $p^* = \frac{\sqrt{17}-1}{4}$*.*

The fact that the strategy profile in Example 2 is an equilibrium has been proved directly from the first principles of maximisation in Dickson and Tonin, 2018 (Example 3 in their paper). We can check these values with our equilibrium condition easily. To see this, let's take the condition just for agent 1 (or 2) in this example. The allocation for agent 1 at equilibrium is given by $x = \frac{\sqrt{17}-1}{2}$ and $y = \sqrt{17} - 3$. At these values, our ratio $\Delta_1(q,b)$ becomes:

$\Delta_1(q,b) = \frac{4(\sqrt{17}-3)(\sqrt{17}-1)}{8(7-\sqrt{17})} = \frac{(\sqrt{17}-1)(\sqrt{17}-3)(\sqrt{17}+3)}{2(7-\sqrt{17})(\sqrt{17}+3)} = \frac{\sqrt{17}-1}{\sqrt{17}+1} = \frac{(\sqrt{17}-1)^2}{16}$,

which is equal to $(\frac{\sqrt{17}-1}{4})^2 = p^2$, confirming our condition obtained in Theorem 1.

4 APPENDIX (PROOFS)

We collect the proofs of our results in this section.

Proof of Lemma 1. Given a market game with the market $\Xi = (N, X, E, U)$, if (q, b) is a trading equilibrium, then for each $i \in N$, (q_i, b_i) (given $(q_j, b_j)_{j\in N\setminus\{i\}}$) maximises $U_i(x_i(q, b))$ subject to $q_{ij} \in [0, e_{ij}]$, $b_{ij} \geq 0$ for $j = 1, \ldots, l$, and $\sum_{k=1}^{l} b_{ik} \leq e_{im}$.

In general, we have $\frac{\partial x_{ij}}{\partial q_{ij}} = -(1 - \frac{b_{ij}}{B(j)}) \leq 0$, $p_j \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial x_{ij}}{\partial q_{ij}} = 0$, $\frac{\partial x_{im}}{\partial b_{ij}} = -(1 - \frac{q_{ij}}{Q(j)}) \leq 0$ and $p_j \frac{\partial x_{im}}{\partial b_{ij}} + \frac{\partial x_{im}}{\partial q_{ij}} = 0$.

Define $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{il}) \in \Re^l_+$, $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{il}) \in \Re^l_+$, $\beta_i = (\beta_{i1}, \ldots, \beta_{il}) \in \Re^l_+$ and $\delta_i \in \Re_+$. The Lagrangian function for the optimisation problem of traders $i \in N$ is the following:

$$L(b, q, \lambda_i, \gamma_i, \beta_i, \delta_i) = U_i(x_i(q, b)) + \sum_{k=1}^{l} \lambda_{ik} q_{ik} + \sum_{k=1}^{l} \gamma_{ik}\left(e_{ik} - q_{ik}\right) + \sum_{k=1}^{l} \beta_{ik} b_{ik} + \delta_i \left(e_{im} - \sum_{k=1}^{l} b_{ik}\right). \quad (5)$$

The Kuhn-Tucker conditions are the following:

$$\frac{\partial L}{\partial q_{ij}} = \frac{\partial U_i}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial q_{ij}} + \frac{\partial U_i}{\partial x_{im}} \frac{\partial x_{im}}{\partial q_{ij}} + \lambda_{ij} - \gamma_{ij} \leq 0 \text{ and } q_{ij} \frac{\partial L}{\partial q_{ij}} = 0, \text{ for each good } j, \quad (6)$$

$$\lambda_{ij} \geq 0, q_{ij} \geq 0 \text{ and } \lambda_{ij} q_{ij} = 0, \text{ for each multiplier } \lambda_{ij}, \text{ given } i, \quad (7)$$

$$\gamma_{ij} \geq 0, e_{ij} \geq q_{ij} \text{ and } \gamma_{ij}(e_{ij} - q_{ij}) = 0, \text{ for each multiplier } \gamma_{ij}, \text{ given } i, \quad (8)$$

$$\frac{\partial L}{\partial b_{ij}} = \frac{\partial U_i}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial U_i}{\partial x_{im}} \frac{\partial x_{im}}{\partial b_{ij}} + \beta_{ij} - \delta_i \leq 0 \text{ and } b_{ij} \frac{\partial L}{\partial b_{ij}} = 0, \text{ for each good } j, \quad (9)$$

$$\beta_{ij} \geq 0, b_{ij} \geq 0 \text{ and } \beta_{ij} b_{ij} = 0, \text{ for each multiplier } \beta_{ij} \text{ given } i, \text{ and} \quad (10)$$

$$\delta_i \geq 0,\ e_{im} \geq \sum_{k=1}^{l} b_{ik} \text{ and } \delta_i \left(e_{im} - \sum_{k=1}^{l} b_{ik}\right) = 0 \text{ for the multiplier } \delta_i. \quad (11)$$

From the first part of condition (6) it follows that for each good j,

$$-\left(\frac{B_{-i}(j)}{B(j)}\right) \frac{\partial U_i}{\partial x_{ij}} + p_j \left(\frac{Q_{-i}(j)}{Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} + \lambda_{ij} - \gamma_{ij} \leq 0, \quad (12)$$

and from the first part of condition (9) it follows that for each good j,

$$\frac{1}{p_j} \left(\frac{B_{-i}(j)}{B(j)}\right) \frac{\partial U_i}{\partial x_{ij}} - \left(\frac{Q_{-i}(j)}{Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} + \beta_{ij} - \delta_i \leq 0. \quad (13)$$

From (12) it follows that

$$\left(\frac{Q_{-i}(j)}{p_j Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} \left(p_j^2 - \Delta_{ij}(q, b)\right) \leq -\lambda_{ij} + \gamma_{ij}. \quad (14)$$

If $q_{ij} \in [0, e_{ij})$, then given $\gamma_{ij}(e_{ij} - q_{ij}) = 0$ and $e_{ij} > q_{ij}$ we get $\gamma_{ij} = 0$. Hence, using $\gamma_{ij} = 0$, $[Q_{-i}(j)]/[p_j Q(j)] > 0$ and $\frac{\partial U_i}{\partial x_{im}} > 0$, from (14) we get the following:

(R1) If $q_{ij} \in [0, e_{ij})$, then $p_j^2 \leq \Delta_{ij}(q, b)$.

Pre-multiplying (12) by q_{ij} and using $\lambda_{ij} q_{ij} = 0$ and $q_{ij}\, [\partial L/\partial q_{ij}] = 0$, we get,

$$q_{ij} \left[-\left(\frac{B_{-i}(j)}{B(j)}\right) \frac{\partial U_i}{\partial x_{ij}} + p_j \left(\frac{Q_{-i}(j)}{Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} - \gamma_{ij} \right] = 0. \tag{15}$$

If $q_{ij} > 0$, then we have

$$\left(\frac{Q_{-i}(j)}{p_j Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} \left(p_j^2 - \Delta_{ij}(q, b)\right) = \gamma_{ij}. \tag{16}$$

Since $\gamma_{ij} \geq 0$, $Q_{-i}(j)/Q(j) > 0$ and $\frac{\partial U_i}{\partial x_{im}} > 0$, (16) implies $p_j^2 \geq \Delta_{ij}(q, b)$. Hence, we have,

(R2) If $q_{ij} > 0$, then $p_j^2 \geq \Delta_{ij}(q, b)$.

Combining $(R1)$ and $(R2)$, we get Lemma 1. ■

Proof of Lemma 2. We prove Lemma 2 following the proof of Lemma 1.

From (13) it follows that

$$\left(\frac{Q_{-i}(j)}{p_j^2 Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} \left(\Delta_{ij}(q, b) - p_j^2\right) \leq -\beta_{ij} + \delta_i. \tag{17}$$

If $\sum_{k=1}^{l} b_{ik} < e_{im}$, then given $\delta_i(e_{im} - \sum_{k=1}^{l} b_{ik}) = 0$ and $\sum_{k=1}^{l} b_{ik} < e_{im}$ we get $\delta_i = 0$. Hence, using $\delta_i = 0$, $[Q_{-i}(j)]/[p_j^2 Q(j)] > 0$ and $\frac{\partial U_i}{\partial x_{im}} > 0$, from (17) we get Lemma 2(a).

Pre-multiplying (13) by b_{ij} and using $\beta_{ij} b_{ij} = 0$ and using $b_{ij}\, [\partial L/\partial b_{ij}] = 0$ we get,

$$b_{ij} \left[\left(\frac{1}{p_j}\right) \left(\frac{B_{-i}(j)}{B(j)}\right) \frac{\partial U_i}{\partial x_{ij}} - \left(\frac{Q_{-i}(j)}{Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} - \delta_i \right] = 0. \tag{18}$$

If $b_{ij} > 0$, we have,

$$\left(\frac{1}{p_j}\right) \left(\frac{B_{-i}(j)}{B(j)}\right) \frac{\partial U_i}{\partial x_{ij}} - \left(\frac{Q_{-i}(j)}{Q(j)}\right) \frac{\partial U_i}{\partial x_{im}} - \delta_i = 0. \tag{19}$$

From (19) it follows that if $b_{ij} > 0$, then

$$\Delta_{ij}(q, b) = p_j^2 + \frac{p_j^2 \delta_i}{\left(\frac{Q_{-i}(j)}{Q(j)}\right)\left(\frac{\partial U_i}{\partial x_{im}}\right)}. \tag{20}$$

Since $\delta_i \geq 0$, $Q_{-i}(j)/Q(j) > 0$ and $\frac{\partial U_i}{\partial x_{im}} > 0$, from (20) we get $p_j^2 \leq \Delta_{ij}(q, b)$. Hence, we have established Lemma 2(b). ■

Proof of Proposition 1. To complete the proof of this proposition, we need to check why pseudo-concavity is sufficient for the conditions $(K1)$-$(K5)$. One can verify that for any $i \in N$, any trading decision $(b_j, q_j)_{j \in N \setminus \{i\}} \in \Re_{++}^{2l \times |N|}$ and any commodity $j \in \{1 \ldots, l\}$, we have the following restrictions on the partial derivatives of the associated function $V_i : \Re_{++}^{2l} \to \Re$.

(i) $\frac{\partial V_i(b_i,q_i)}{\partial b_{ij}} = \left(\frac{B_{-i}(j)}{p_jB(j)}\right)\frac{\partial U_i}{\partial x_{ij}} - \left(\frac{Q_{-i}(j)}{Q(j)}\right)\frac{\partial U_i}{\partial x_{im}} = \left(\frac{Q_{-i}(j)}{p_j^2Q(j)}\right)\frac{\partial U_i}{\partial x_{im}}[\Delta_{ij} - p_j^2]$ and

(ii) $\frac{\partial V_i(b_i,q_i)}{\partial q_{ij}} = -\left(\frac{B_{-i}(j)}{B(j)}\right)\frac{\partial U_i}{\partial x_{ij}} + \left(\frac{p_jQ_{-i}(j)}{Q(j)}\right)\frac{\partial U_i}{\partial x_{im}} = \left(\frac{Q_{-i}(j)}{p_jQ(j)}\right)\frac{\partial U_i}{\partial x_{im}}[p_j^2 - \Delta_{ij}]$.

We also have the restriction (iii) below (that follows immediately from (i) and (ii)):

(iii) $p_j\frac{\partial V_i(b_i,q_i)}{\partial b_{ij}} + \frac{\partial V_i(b_i,q_i)}{\partial q_{ij}} = 0$.

For any $k \in \{1, \ldots, l\}$ and any pair of trading decisions $(b_i', q_i'), (b_i, q_i)) \in \Re^{2l}_{++}$ for trader i, define $A(k) := [(b_{ik}' - b_{ik}) - p_k(q_{ik}' - q_{ik})]\,[\Delta_{ik} - p_k^2]$. From (i) and (iii) and by using $\frac{\partial U_i}{\partial x_{im}} > 0$ in (2), we get the following condition:

$$\text{If } \sum_{k=1}^{l} A(k)\left(\frac{Q_{-i}(k)}{p_k^2Q(k)}\right) \leq 0, \text{ then } V_i(b_i', q_i') \leq V_i(b_i, q_i). \tag{21}$$

Suppose (b_i, q_i) satisfies $(K1)$-$(K5)$, for all $k \in \{1, \ldots, n\}$. Consider any $(b_i', q_i')(\neq (b_i, q_i))$ and then consider the sum $\sum_{k=1}^{l} A(k)[Q_{-i}(k)/\{p_k^2Q(k)\}]$.

If $p_k^2 = \Delta_{ik}$, then $A(k) = 0$.

If $q_{ik} = 0$, $\sum_{k=1}^{l} b_{ik} = e_{im}$ and $p_k^2 \leq \Delta_{ik}$ holds, then $b_{ik}' \leq b_{ik}$ and $q_{ik}' \geq q_{ik}$ implying $A(k) \leq 0$.

Finally, if $q_{ik} = e_{ik}$, $b_{ik} = 0$ and $p_k^2 \geq \Delta_{ik}$ holds, then $b_{ik}' \geq b_{ik}$ and $q_{ik}' \leq q_{ik}$ implying $A(k) \leq 0$.

Hence, if for trader i, (b_i, q_i) satisfies the conditions $(K1)$-$(K5)$, then it follows that for any $(b_i', q_i')(\neq (b_i, q_i))$, we have $\sum_{k=1}^{l} A(k)\left(\frac{Q_{-i}(k)}{p_k^2Q(k)}\right) \leq 0$ since $\left(\frac{Q_{-i}(k)}{p_k^2Q(k)}\right) > 0$ and $A(k) = [(b_{ik}' - b_{ik}) - p_k(q_{ik}' - q_{ik})]\,[\Delta_{ik} - p_k^2] \leq 0$ for all k. By pseudo-concavity, we then have $V_i(b_i, q_i) \geq V_i(b_i', q_i')$ implying that the conditions $(K1)$-$(K5)$ are also sufficient. ■

Proof of Theorem 1. Specifically, from $(K1)$, $(K2)$ and $(K4)$, we get the first condition and (c) of the second condition in Theorem 1; from $(K3)$, we get (a) of the second condition and from $(K4)$, we get (b) of the second condition in Theorem 1. ■

Proof of Theorem 2. If (b, q) is a trading equilibrium for a buy or sell strategic market game, then for each $i \in N$, (b_i, q_i) (given $((b_j, q_j)_{j\in N\setminus\{i\}})$ maximises $U_i(x_i(b, q))$ subject to $q_{ij} \in [0, e_{ij}]$, $b_{ij} \geq 0$ for $j = 1, \ldots, l$, $\sum_{k=1}^{l} b_{ik} \leq e_{im}$ and $b_{ij}q_{ij} = 0$. Define $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{il}) \in \Re^l_+$, $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{il}) \in \Re^l_+$, $\beta_i = (\beta_{i1}, \ldots, \beta_{il}) \in \Re^l_+$, $\delta_i \in \Re_+$ and $\kappa_i \in \Re_+$. Given the Lagrangian function $L(b, q, \lambda_i, \gamma_i, \beta_i, \delta_i)$ from (5) for the strategic market game, the Lagrangian function $\bar{L}(.)$ for the optimisation problem of traders $i \in N$ in the buy or sell strategic market game is the following:

$$\bar{L}(b, q, \lambda_i, \gamma_i, \beta_i, \delta_i, \kappa_i) = L(b, q, \lambda_i, \gamma_i, \beta_i, \delta_i) + \kappa_i b_{ij} q_{ij}. \tag{22}$$

The new added constraint for the Lagrangian function $\bar{L}(b, q, \lambda_i, \gamma_i, \beta_i, \delta_i, \kappa_i)$ is that $\kappa_i \geq 0$, $b_{ij} \geq 0$, $q_{ij} \geq 0$ and $\kappa_i b_{ij} q_{ij} = 0$ for the new multiplier κ_i. The proof of this theorem becomes easy from the following observations.

1. $\frac{\partial \bar{L}}{\partial q_{ij}} = \frac{\partial L}{\partial q_{ij}} + \kappa_i b_{ij}$.

2. $q_{ij}\frac{\partial \bar{L}}{\partial q_{ij}} = q_{ij}\frac{\partial L}{\partial q_{ij}} + \kappa_i b_{ij} q_{ij} = q_{ij}\frac{\partial L}{\partial q_{ij}} = 0$ (since $\kappa_i b_{ij} q_{ij} = 0$).

3. $\frac{\partial \bar{L}}{\partial b_{ij}} = \frac{\partial L}{\partial b_{ij}} + \kappa_i q_{ij}$.

4. $b_{ij}\frac{\partial \bar{L}}{\partial b_{ij}} = b_{ij}\frac{\partial L}{\partial b_{ij}} + \kappa_i b_{ij} q_{ij} = b_{ij}\frac{\partial L}{\partial b_{ij}} = 0$.

Hence, by using arguments similar to the ones used earlier, one can also prove lemmata corresponding to Lemma 1 and Lemma 2 for this buy or sell strategic market game. Specifically, one can show the following:

(a) If $q_{ij} \in (0, e_{ij})$, then $p_j^2 = \Delta_{ij}(b, q)$.

(b) If $q_{ij} = 0$, then $p_j^2 \leq \Delta_{ij}(b, q)$.

(c) If $q_{ij} = e_{ij}$, then $p_j^2 \geq \Delta_{ij}(b, q)$.

(d) If $\sum_{k=1}^{l} b_{ik} < e_{im}$, then $p_j^2 \geq \Delta_{ij}(b, q)$.

From (a)-(d), it follows that for any $i \in N$ and any $j = 1, \ldots, l$, the following conditions hold:

(k1) If $q_{ij} \in (0, e_{ij})$ and $b_{ij} = 0$, then $p_j^2 = \Delta_{ij}(b, q)$.

(k2) If $q_{ij} = 0$ and $\sum_{k=1}^{l} b_{ik} < e_{im}$, then $p_j^2 = \Delta_{ij}(b, q)$.

(k3) If $q_{ij} = 0$ and $\sum_{k=1}^{l} b_{ik} = e_{im}$, then $p_j^2 \leq \Delta_{ij}(b, q)$.

(k4) If $q_{ij} = e_{ij}$ and $b_{ij} = 0$, then $p_j^2 \geq \Delta_{ij}(b, q)$.

Specifically, from (a) we get condition $(k1)$ using the buy or sell restriction. From (b) and (d) we get condition $(k2)$. Given condition $(k2)$, from (b) we also get condition $(k3)$. From (c), we get condition $(k4)$ using the buy or sell restriction. Finally, from $(k1)$ and $(k2)$, we get $(kt1)$ and $(kt2)(c)$ in the statement of Theorem 2; we get $(kt2)(a)$ from $(k3)$ and, from $(k4)$, we get $(kt2)(b)$.

Given our assumption that the utility function $U_i(x_i(b, q))$ of each trader $i \in N$ is continuously differentiable and pseudo-concave and given that the constraints are quasi-convex, the Kuhn-Tucker conditions are both necessary and sufficient to characterise the equilibrium outcomes. The arguments for checking why pseudo-concavity is sufficient for the Kuhn-Tucker conditions $(k1)$-$(k4)$ above is similar to the arguments used in the proof of Theorem 1 (and Proposition 1) and hence is omitted. ■

Proof of Proposition 2. In a market game with the market Ξ_2^{C-D}, the equilibrium consumptions of the two goods x and y for the two agents are:

- $x_1 = \omega_x^1 + b_1/p - q_1$ so that $x_1 = \omega_x^1 + b_1 - q_1 = \omega_x^1 + k$,
- $y_1 = \omega_y^1 - b_1 + pq_1$ so that $y_1 = \omega_y^1 - b_1 + q_1 = \omega_y^1 - k$,
- $x_2 = \omega_x^2 + b_2/p - q_2$ so that $x_2 = \omega_x^2 + b_2 - q_2 = \omega_x^2 - k$, and
- $y_2 = \omega_y^2 - b_2 + pq_2$ so that $y_2 = \omega_y^2 - b_2 + q_2 = \omega_y^2 + k$.

Given the Cobb-Douglas utility function, for any $i \in \{1, 2\}$, $MRS_{x,y}^i = \frac{\partial U_i(x_i,y_i)}{\partial x_i} / \frac{\partial U_i(x_i,y_i)}{\partial y_i} = (\alpha_i/(1-\alpha_i))(y_i/x_i)$.

Equilibrium condition of the market game is $\left(\frac{b_2}{q_2}\right) MRS_{x,y}^1 = \left(\frac{b_1}{q_1}\right) MRS_{x,y}^2 = p^2 = 1$. From the equilibrium condition, we have the following implications.

For agent 1, we have
$\left(\frac{b_2}{q_2}\right)\left(\frac{\alpha_1}{1-\alpha_1}\right)\left(\frac{\omega_y^1-k}{\omega_x^1+k}\right) = 1 \Rightarrow \alpha_1 b_2(\omega_y^1 - k) = (1-\alpha_1)q_2(\omega_x^1 + k)$
$\Rightarrow \alpha_1 b_2(\omega_y^1 - k) = (1-\alpha_1)(b_2+k)(\omega_x^1 + k)$
$\Rightarrow \alpha_1 b_2\omega_y^1 = (1-\alpha_1)(b_2+k)\omega_x^1 + k[(1-\alpha_1)(b_2+k) + \alpha_1 b_2]$
$\Rightarrow \alpha_1 b_2\omega_y^1 = (1-\alpha_1)(b_2+k)\omega_x^1 + k[b_2 + (1-\alpha_1)k]$
$\Rightarrow \alpha_1 b_2\omega_y^1 = (1-\alpha_1)q_2\omega_x^1 + k[\alpha_1 b_2 + (1-\alpha_1)q_2]$.

Therefore, for agent 1, we have (3) as a restriction for an interior equilibrium.

Similarly, for agent 2, we have
$\left(\frac{b_1}{q_1}\right)\left(\frac{\alpha_2}{1-\alpha_2}\right)\left(\frac{\omega_y^2+k}{\omega_x^2-k}\right) = 1 \Rightarrow \alpha_2 b_1(\omega_y^2 + k) = (1-\alpha_2)q_1(\omega_x^2 - k)$
$\Rightarrow \alpha_2 b_1(\omega_y^2 + k) = (1-\alpha_2)(b_1-k)(\omega_x^2 - k)$
$\Rightarrow (1-\alpha_2)(b_1-k)\omega_x^2 = \alpha_2 b_1\omega_y^2 + k[(1-\alpha_2)(b_1-k) + \alpha_2 b_1]$
$\Rightarrow (1-\alpha_2)(b_1-k)\omega_x^2 = \alpha_2 b_1\omega_y^2 + +k[b_1 - (1-\alpha_2)k]$
$\Rightarrow (1-\alpha_2)q_1\omega_x^2 = \alpha_2 b_1\omega_y^2 + +k[\alpha_2 b_1 + (1-\alpha_2)q_1]$.

Thus, for agent 2, (4) is a restriction for an interior equilibrium. ■

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