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DCC-HEAVY: a multivariate GARCH model with realized measures of variance and correlation

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Abstract

This paper proposes a new class of multivariate volatility model that utilising high-frequency data. We call this model the DCC-HEAVY model as key ingredients are the Engle (2002) DCC model and Shephard and Sheppard (2012) HEAVY model. We discuss the models' dynamics and highlight their differences from DCC-GARCH models. Specifically, the dynamics of conditional variances are driven by the lagged realized variances, while the dynamics of conditional correlations are driven by the lagged realized correlations in the DCC-HEAVY model. The new model removes well known asymptotic bias in DCC-GARCH model estimation and has more desirable asymptotic properties. We also derive a Quasi-maximum likelihood estimation and provide closed-form formulas for multi-step forecasts. Empirical results suggest that the DCC-HEAVY model outperforms the DCC-GARCH model in and out-of-sample.

Keywords: HEAVY model, Multivariate volatility, High-frequency data, Forecasting, Wishart distribution

JEL Classification: C32, C58, G17

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1 Introduction

Multivariate volatility modeling plays a critical role in the areas of risk management, portfolio optimization, and asset pricing. Therefore, understanding and forecasting the temporal and cross-sectional dependence in elements of the covariance matrix are of vital importance in financial econometrics. The most popular approach is the multivariate generalized autoregressive conditional heteroscedasticity (GARCH) models (reviewed by Bauwens et al., 2006), which treats the conditional covariance matrix of returns as a deterministic function of past returns.

More recently, the increasing availability of intraday data has led to the introduction of new types of volatility models that include so-called "high-frequency-based volatility model". These new models lead to more accurate measurements and forecasts of the conditional (co)variance of daily financial returns. Examples of such models in the univariate case are the multiplicative error model (MEM) (Engle and Gallo 2006), the HEAVY (high-frequency-based volatility) model (Shephard and Sheppard 2010), and the Realized GARCH model (Hansen et al. (2012). In the multivariate context, Noureldin et al. (2012) develop a multivariate HEAVY model, which consists of dynamic specifications for conditional covariance of returns and realized measures of the covariance matrix. In particular, they show that the dynamics of conditional covariance are driven by the lagged realized covariance rather that the lagged outer product of daily returns. The related multivariate HEAVY models are Jin and Maheu (2013) and Opschoor et al.(2017), who developed dynamic component models of returns and realized covariance matrices based on Wishart distributions.[1] The multivariate HEAVY model leads to more accurate measurements and forecasts of the conditional covariance matrix and becomes a popular approach in the multivariate volatility modeling. The multivariate HEAVY models (i.e. Noureldin et al., 2012 and Opschoor et al., 2017) usually adopt a scalar BEKK type parameterization to avoid the curse of dimensionality. This scalar model implies that the conditional variances and covariances all follow the same dynamic pattern. This is restrictive but reduces the number of parameters enormously.

In this paper, we propose an alternative way to specify the conditional covariance matrix in the multivariate HEAVY framework. We adopt a DCC

[1] The related autoregressive Wishard model for the covariance matrix are Gourieroux et al. (2009),Chiriac and Voev (2011) and Golosnoy, et al. (2012), although the authors apply it to realized covariance matrices, discarding the daily return observations.

type parameterization. The DCC type parameterization, which is an equal or more popular specification than BEKK type parameterization, has attracted less attention. However It is well known the DCC model is applicable to large conditional variance-covariance matrix estimation, without imposing a scalars restriction on the parameters, as the DCC type model is a two-step estimator. It is therefore interesting to study the DCC type parameterization in the multivariate HEAVY framework.

The new model we propose is called DCC-HEAVY model as key ingredients are the DCC model and HEAVY model. Specifically, we representation the conditional covariance matrix of returns in terms of the product of the corresponding diagonal matrix of conditional standard deviation and conditional correlation matrix. Different from DCC-GARCH models, the dynamics of conditional variances(which is the square of standard deviation) are driven by the lagged realized variances, while the dynamics of conditional correlations are driven by the lagged realized correlations.

We discuss the models' dynamics and highlight their differences from DCC-GARCH models. In particular, we show that the new model removes well known asymptotic bias in DCC-GARCH model estimation and has more desirable asymptotic properties. We also unify the DCC-HEAVY, DCC-GARCH and DCCX-GARCHX[2] in one representation and express them in a vector multiplicative error form. The statioanry condition and closed-form formulas for multi-step forecasts are derived.

The DCC-HEAVY model is a two-step estimator of conditional variances and correlations. In the first step the mean equation of each asset in the sample, nested in a univariate HEAVY model of its conditional variance, is estimated. Then these univariate variance estimates are used to standardise the return innovations for each asset. In the second step, a model of the first moments of the standardised return innovations nested in a scalar multivariate GARCH model of conditional second moments is estimated. Engle (2002) show that this two steps procedure produces consistent maximum likelihood parameter estimates. The two-step approach makes the estimation much simple, and hence the DCC model is applicable to large conditional variance-covariance matrix estimation, without imposing a scalars restriction on the parameters.

In an empirical application, we apply the model to 10 most liquid stocks in the Dow Jones Industrial Average (DJIA) index. Like the univariate HEAVY

[2] 'X' denotes the exogeneous variables. See section 3.1 for details.

model, we find that the effects of the lagged returns squared are insignificant when lagged realized variances are included in the conditinoal variance equations, which implies that the dynamics of conditional variance are driven by the lagged realized variance only. Likewise, we find that the effect of the lagged out product of return innovation is insignificant when lagged realized correlation is included in the conditinoal correlation equation, which implies that the dynamics of conditional correlation are driven by the lagged realized correlation only. Moreover, our results suggest that the DCC-HEAVY model outperforms the DCC-GARCH model both in and out-of-sample. It fits with the data better and provides better out-of-sample forecasts. The out-of-sample forecast gains comparing with theDCC-GARCH model is substatial. The gains ranges from 10% to 40%.

The remainder of the paper is organized as follows. Section 2 introduces the DCC-HEAVY model with a comparison to the DCC-GARCH model. Section 3 analysis the model properties in details, including multiplicative error representation, multi-step forecasts and model extension. Section 4 discusses the estimation and inference. Section 5 presents the results of our empirical analysis, along with various statistical in- and out-of-sample forecast comparisons. Section 6 concludes. The additional Appendix to this paper includes additional empirical results.

2 The DCC-HEAVY Model

2.1 Multivariate HEAVY framework

Noureldin et al. (2012) extend the HEAVY model in a multivariate version. Follow their definition, we define r_t to be a $(k \times 1)$ vector of daily returns and $r_{j,t}$ be the jth intra-daily vector of returns on a day t where $j = 1, 2, ..., m$. Assuming, for instance, 24-hour trading means $m = 1440$. The outer product of daily returns is the $(k \times k)$ matrix denoted by $r_t r_t'$. The realized covariance measure on the day t is a $(k \times k)$ matrix denoted by RC_t. One example of RC_t which is used in Noureldin et al. (2012) and this paper is the realized covariance matrix defined as

$$RC_t = \sum_{j=1}^{m} r_{j,t} r_{j,t}'$$

An alternative is to use a noise-robust estimator such as the realized kernel of Barndorff-Nielsen et al. (2008, 2011). We then define the realized correlation on the day t as a $(k \times k)$ matrix denoted by RL_t, such that $RL_t = di\alpha g(RC_t)^{-1/2} RC_t di\alpha g(RC_t)^{-1/2}$, with unit on the main diagonal. The realized variance on the day t is a $(k \times 1)$ vector denoted by v_t, which only includes the diagonal elements in RC_t.

The multivariate HEAVY model is the two-equation system

$$\begin{aligned} E(r_t r_t' | \mathcal{F}_{t-1}) &: = H_t \\ E(RC_t | \mathcal{F}_{t-1}) &: = M_t \end{aligned}$$

where $E(r_t | \mathcal{F}_{t-1}) = 0$ is assumed for simplicity, so that H_t is the conditional covariance matrix of daily return and M_t is the conditional mean of the realized covariance matrix.

For the specification of the dynamics of H_t and M_t, Noureldin et al. (2012) adopt the BEKK-type model to ensure that the conditional covariance matrix is positive semidefinite. The key feature of this BEKK-HEAVY model is that the conditional covariance H_t is a function of lagged realized covariance RC_t rather than the outer product of daily return $r_t r_t'$. The unrestricted BEKK-type parameterization has $O\left(k^2\right)$ parameters. In applied work, the parameter matrices are usually imposed to be scalars of diagonal matrices to avoid the curse of dimensionality.

2.2 DCC-HEAVY Model

An alternative specification for the condition covariance matrix is the conditional correlation (DCC-)GARCH type model, proposed by Engle (2002). The idea is to decompose the conditional covariance matrix of returns in terms of the product of the corresponding diagonal matrix of conditional standard deviation and conditional correlation matrix. Let's rewrite the return vector r_t by the relation

$$r_t = u_t \odot h_t^{1/2},\, u_t \sim i.i.d\ N(0, R_t) \tag{1}$$

where $\odot$ denotes the Hadamard (element by element) product; h_t is the conditional variance of returns and R_t is the conditinal correlation matrix of returns. R_t is a symmetric positive definite correlation matrix with unit diagonal elements.

The conditional covariance matrix, H_t , can be decomposed as:

$$H_t = di\alpha g[h_t^{1/2}] R_t di\alpha g[h_t^{1/2}].) \tag{2}$$

The specification for the conditional covariance consists with the specifications of conditional variance h_t and conditional correlation R_t. The dynamics of the conditional variance can be expressed as HEAVY-r (Shephard and Sheppard, 2010):

$$h_t{=}\omega_h + A_h v_{t-1} + B_h h_{t-1} \tag{3}$$

where ω_h is a $(k \times 1)$ vector, A_h and B_h are $(k \times k)$ a diagonal matrix [3].

The dynamics of the conditional correlation can be expressed similarly as:

$$R_t = \tilde{R} + \alpha_r RL_{t-1} + \beta_r R_{t-1} \tag{4}$$

where $\tilde{R} = (1 - \beta_r)\ \bar{R} - \alpha_r \bar{P}$ is a $k \times k$ matrix; $\bar{R}$ equals the unconditional correlation matrix of u_t; $\bar{P}$ equals the unconditional mean matrix of RL_t; α_q, β_q are non-negative scalar parameters satisfying $\beta_\rho < 1$. The elements of $\bar{R}$ and $\bar{P}$ can be set to their empirical counterpart to render the estimation simpler. There are only two parameters (α_q and β_q) that are needed to be estimated.

Since RL_t has unit diagonal elements, R_t is a well-defined correlation matrix for all t if the initial matrix R_0 is a correlation matrix. Matrix $\tilde{R}$ satisfies the constraints of a correlation matrix, i.e. positive definite symmetric with unit diagonal elements. Hence, reparameterising R_t as in the DCC-GARCH model is not needed.

Eq (2), (3) and (4) forms the DCC-HEAVY model.

To close the DCC-HEAVY model, we need to specify the dynamics for the realized covariance matrix. Following Bauwens et al. (2012), the realized covariance matrix can be specified as a DCC type model such that

$$E(RC_t|\mathcal{F}_{t-1}) := M_t$$

$$M_t = di\alpha g[m_t^{1/2}] P_t di\alpha g[m_t^{1/2}]. \tag{5}$$

where m_t is the conditional mean of realized variance and P_t is the conditional mean of the realized correlation matrix.

The realized variance m_t follows a HEAVY-RM (or MEM) structure:

$$m_t{=}\omega_m + A_m v_{t-1} + B_m m_{t-1} \tag{6}$$

[3] A_h can be full $k \times k$ matrix to allow spillover effects, but B_h is restricted to be a diagnol matrix. This specification enables estimating the k-dimentional variance model eqation-by-equation.

where ω_m is a $(k \times 1)$ vector, A_m and B_m are $(k \times k)$ matrix.

The realized correlation matrix P_t is defined in the following way:

$$P_t = \tilde{P} + \alpha_p RL_t + \beta_p P_{t-1} \tag{7}$$

where $\tilde{P} = (1 - \alpha_p - \beta_p)\bar{P}$ with $\bar{P}$ an $k \times k$ unconditional mean matrix of RL_t, and α_p, β_p are non-negative scalar parameters satisfying $\alpha_p + \beta_p < 1$. The elements of $\bar{P}$ can be set to their empirical counterpart to render the estimation simpler[4].

The DCC-HEAVY model can be estimated in two steps, and a QML interpretation is given to each step in section 3.

Remark 1 *The DCC-HEAVY model can be simplified to a CCC-HEAVY model if R_t is constant. The CCC-HEAVY model consistent the following two equations:*

$$\begin{aligned} r_t &= u_t \odot h_t^{1/2},\ u_t \sim i.i.d(0, R) & (8) \\ h_t &= \omega_h + A_h v_{t-1} + B_h h_{t-1} & \\ v_t &= m_t \odot \varepsilon_t^v\ , \quad \varepsilon_t^v \sim i.i.d(I, \Sigma_v) & (9) \\ m_t &= \omega_m + A_m v_{t-1} + B_m m_{t-1} & \end{aligned}$$

The correlation matrix R is constant, which can be estimated jointly with other parameters in h_t equation.

2.3 Comparison with the DCC-GARCH model

DCC-GARCH model (Engle 2002) for the conditional covariance matrix consists with the specifications of conditional variance h_t

$$h_t = \omega_h + A_h r_{t-1} + B_h h_{t-1} \tag{10}$$

where ω_h is a $(k \times 1)$ vector, A_h and B_h are $(k \times k)$ diagonal matrix, and specification of conditional correlation R_t

$$R_t = \{diag(Q_t)\}^{-1/2} Q_t \{diag(Q_t)\}^{-1/2} \tag{11}$$

[4] $E(RL_t)$ is not equal to the uncontitional correlation matrix P_t,due to the non-linearity of the the transformation from covariance to correlation. However, Bauwens et al. (2012) shows that for large RC_t (RC_t is caculated from large enough number of high-frequency returns), $\bar{P}$ should be equal to $E(RL_t)$.

where the $k \times k$ symmetric positive definite matrix $Q_t = (q_{ij,t})$ is given by[5]:

$$Q_t = (1 - \alpha_q - \beta_q)\bar{Q} + \alpha_q[u_{t-1}u'_{t-1}] + \beta_q Q_{t-1} \tag{12}$$

where α_q, β_q are non-negative scalar parameters satisfying $\alpha_q + \beta_q < 1$ and $\bar{Q}$ is the $k \times k$ matrix.

Note that the parameterising $R_t = \{diag(Q_t)\}^{-1/2} Q_t \{diag(Q_t)\}^{-1/2}$ is to guarantee R_t a well-defined correlation matrix (positive defined symmetric matrix with unit diagonal elements). Q_t itself does not have this property since $u_{t-1}u'_{t-1}$ is not a symmetric matrix with non-unit diagonal elements. However, this parameterisation results in two issues in the DCC model estimates and forecasts.

First, Engle (2002) assumes that $\bar{Q} \simeq \bar{R}$, where $\bar{R}$ is the unconditional correlation of u_t. The elements $\bar{Q}$ are set to their empirical counterpart to render the estimation simpler. However due to the non-linearity of the transformation from R_t to Q_t[6], the term $\bar{Q}$ is not the unconditional correlation of u_t, which is also acknowledged in Engle and Sheppard (2001) and Aielli (2013). Therefore, $\bar{Q}$ is a biased estimator and the effect is that the estimates α_q and β_q will be biased as well. These asymptotic biases are due to the fact that $\frac{1}{T}\sum_t u_t u'_u$ does not converge to $\bar{Q}$.

Second, multiple-step ahead forecasting requires $E_t(R_{t+s})$. However, the DCC evolution process is a non-linear process

$$Q_{t+s} = \tilde{Q} + \alpha_q[u_{t+s-1}u'_{t+s-1}] + \beta_q Q_{t+s-1}$$

where $E_t[u_{t+s-1}u'_{t+s-1}] = E_t[R_{t+s-1}]$ and $R_{t+s} = \{diag(Q_t)\}^{-1/2} Q_t \{diag(Q_t)\}^{-1/2}$. Thus, the s-step ahead forecast of the correlation cannot be directly solved forward to provide a convenient method for forecasting. A simple approximation (Engle and Sheppard 2001) to set $E_t[Q_{t+s}] \approx E_t[R_{t+s}]$, then derive forecast as

$$\begin{aligned} R_{t+s|s} &= \bar{Q} + \alpha_q[u_{t+s-1|s}u'_{t+s-1|s}] + \beta_q R_{t+s-1|s} \\ &= \bar{Q} + (\alpha_q + \beta_q) R_{t+s-1} \end{aligned} \tag{13}$$

But this is a biased forecast.

Due to the asymptotic bias, the DCC-GARCH model may result in misleading conclusions. Aielli(2013) shows that the bias is an increasing function

[5] Tse and Tsui (2002) use a different specification in their DCC-GARCH model.

[6] For example,Jason's inequality shows that $E[(x_t)^{-1/2}] \neq [E(x_t)]^{-1/2}$

of the persistence of the correlation process. Engle et al. (2008) and Aielli (2013) suggest to modify the standard DCC in order to correct the asymptotic bias proposes a modified DCC model (cDCC), but additioal issues arisng in estimation, for example, a third step estimation is needed which generate more inefficiency (see Aielli, 2013).

The DCC-HEAVY model is more tractable and does not suffer from the asymptotic bias. It differs with DCC-GARCH model in three ways: 1) the dynamics of conditional variances h_t are driven by the lagged realized volatilities v_{t-1}; 2) the conditional correlation R_t is modelled directly rather than parameterised in a sandwich form as in eq.(11); 3) the dynamics of conditional correlations R_t are driven by the lagged realized correlations RL_{t-1}.

The latter two enables DCC-HEAVY model deriable asymptotic properties. In the correlation matrix equation (eq.(4)), RL_t has unit diagonal elements and is a well-defined correlation matrix for all t and $\tilde{R}$ is a parameter that satisfies the constraints of a correlation matrix. So R_t is a well-defined correlation matrix for all t as long as the initial value R_0 is a correlation matrix. Hence, reparameterising R_t as in the DCC-GARCH model such that $R_t = \{diag(Q_t)\}^{-1/2}Q_t\{diag(Q_t)\}^{-1/2} = Q_t$ is not necessary. As a result, the asymptotic bias due to the nolinear transformation from R_t to Q_t in DCC-GARCH model does not exist in DCC-HEAVY model. Therefore, DCC-HEAVY model is more tractable has a better asympotitc properties than DCC-GARCH model.

3 Representation, Forecasting and Extension

In this section, we look further the properties of DCC-HEAVY model. We first represent the DCC-HEAVY model in a multiplicative error form and discuss its dynamics. We then derive closed-form formulas for multi-step forecasts. Finally, we discuss model extensions by adding asymmetric effects.

3.1 Multiplicative Error Representation

To better understand the dynamics, we express the DCC-HEAVY model in a multiplicative error form.

We begin with the conditional and realized variance equations. Defining $x_t = [r_t^2, v_t]'$ and $\mu_t = [h_t, m_t]'$, where x_t and μ_t is a $(2k \times 1)$ the vector, the vector multiplicative representation for conditional and realized variance equations (eq.(3) and (6)) are

$$
\begin{aligned}
E(x_t|\mathcal{F}_{t-1}) \quad &: \quad = \mu_t \\
\mu_t \quad &= \quad \omega + Ax_{t-1} + B\mu_{t-1}
\end{aligned}
\tag{14}
$$

where

$$
\omega = \begin{bmatrix} \omega_h \\ \omega_m \end{bmatrix}, A = \begin{bmatrix} 0 & A_h \\ 0 & A_m \end{bmatrix}, \ B = \begin{bmatrix} B_h & \\ & B_m \end{bmatrix}.
$$

Note if $A = \begin{bmatrix} A_h & \\ 0 & A_m \end{bmatrix}$, it becomes a DCC-GARCH model; if $A = \begin{bmatrix} A_h & A_{hm} \\ 0 & A_m \end{bmatrix}$, it becomes a DCC-GARCHX model. These are the two models that we will estimate in the empirical analysis for comparison with our DCC-HEAVY model purpose.

Then defining $\mathbf{Y}_t = [u_t u_t', RL_t]'$ and $\mathbf{\Phi}_t = [R_t, P_t]'$, where $\mathbf{Y}_t$ and $\mathbf{\Phi}_t$ is a $(2k \times k)$ matrix, the matrix multiplicative representation for conditional and realized correlation matrix equations (eq.(4) and (7)) are

$$
\begin{aligned}
E(\mathbf{Y}_t|\mathcal{F}_{t-1}) \quad &: \quad = \mathbf{\Phi}_t \\
\mathbf{\Phi}_t \quad &= \quad \boldsymbol{W} + \alpha \mathbf{Y}_{t-1} + \beta \mathbf{\Phi}_{t-1}
\end{aligned}
\tag{15}
$$

where

$$
\boldsymbol{W} = \begin{bmatrix} \tilde{R} \\ \tilde{P} \end{bmatrix}, \alpha = \begin{bmatrix} & \alpha_r \\ 0 & \alpha_p \end{bmatrix}, \ \beta = \begin{bmatrix} \beta_r & \\ & \beta_p \end{bmatrix}
$$

Note if $\alpha = \begin{bmatrix} \alpha_q & \\ 0 & \alpha_p \end{bmatrix}$, it becomes DCC-GARCH model; if $\alpha = \begin{bmatrix} \alpha_q & \alpha_r \\ 0 & \alpha_p \end{bmatrix}$, it becomes DCCX-GARCH model.

Note that R_t is equal to Q_t in DCC(X)-GARCH model, so $E(\mathbf{Y}_t|\mathcal{F}_{t-1}) :\simeq \mathbf{\Phi}_t$. This approximation does affect the multi-step ahead forecasting, as shown in the next subsection.

To summary, the DCC-HEAVY model in multiplicative error form has the following two equations:

1) Conditional and realized variance equations:

$$
\begin{gathered}
E(x_t|\mathcal{F}_{t-1}) := \mu_t \\
\mu_t = \omega + Ax_{t-1} + B\mu_{t-1}
\end{gathered}
\tag{16}
$$

2) Conditional and realized correlation matrix equations:

$$E(\mathbf{Y}_t|\mathcal{F}_{t-1}) := \mathbf{\Phi}_t \\ \mathbf{\Phi}_t = \boldsymbol{W} + \alpha\mathbf{Y}_{t-1} + \beta\mathbf{\Phi}_{t-1} \tag{17}$$

The stationary conditions are given as follows:

$$\max(eig(A+B)) < 1$$

$$\max(eig(\alpha+\beta)) < 1 \tag{18}$$

3.2 Multiple-step ahead Forecasting

We are primarily interested in forecasting the conditional covariance matrix of daily returns, H_t. The s-step ahead forecasts of $H_{t+s|t}$ is:

$$\begin{aligned} E_t[H_{t+s}] &= E_t[di\alpha g(h_{t+s}^{1/2})R_{t+s}di\alpha g(h_{t+s}^{1/2})] \\ &= E_t[di\alpha g(h_{t+s}^{1/2})]E_t[R_{t+s}]E_t[di\alpha g(h_{t+s}^{1/2})] \end{aligned}$$

We need to forecast $E_t[h_{t+s}]$ and $E_t[R_{t+s}]$.

For forecasting of $E_t[h_{t+s}]$ and $E_t[R_{t+s}]$, we use the multiplicative error form, which is more general and valid for both DCC-HEAVY, DCC-GARCH and DCCX-GARCH model. We forecast $E_t(\mu_{t+s})$ and $E_t(\mathbf{\Phi}_{t+s})$, denoted by $\mu_{t+s|t}$ and $\mathbf{\Phi}_{t+s|t}$, respectively.

Let's start with eq.(14) and forecast $\mu_{t+s|t}$.We move steps ahead, $x_{t+s|t}$, $s > 0$ is not known and needs to be substituted with its corresponding conditional expectation $\mu_{t+s|t}$, hence

$$\begin{aligned} \mu_{t+1|t} &= \omega + Ax_t + B\mu_t \\ \mu_{t+s|t} &= \omega + (A+B)\mu_{t+s-1|t} \quad \text{for } s \succeq 2 \end{aligned} \tag{19}$$

which can be solved recursively for any horizon s. A closed form forecasts for $\mu_{t+s|t}$ can be derived as:

$$\mu_{t+s|t} = \tilde{\omega} + C^{s-1}\mu_{t+1|t} \quad \text{for } s \succeq 2$$

where $\tilde{\omega} = (I-C)^{-1}(I-C^{s-1})\omega$ and $C = (A+B)$.

We then derive a similar result for $E_t(\mathbf{\Phi}_{t+s})$ from eq.(15) as following:

$$\begin{aligned}\mathbf{\Phi}_{t+1|t} &= \boldsymbol{W} + \alpha \mathbf{Y}_t + \beta \mathbf{\Phi}_t \\ \mathbf{\Phi}_{t+s|t} &= \boldsymbol{W} + (\alpha + \beta)\mathbf{\Phi}_{t+s-1|t} \quad \text{for } s \succeq 2. \end{aligned} \tag{20}$$

A closed form forecasts for $\phi_{t+s|t}$ can be derived as:

$$\mathbf{\Phi}_{t+s|t} = \tilde{\boldsymbol{W}} + c^{s-1}\mathbf{\Phi}_{t+1|t} \text{ for } s \succeq 2$$

where $\tilde{\boldsymbol{W}} = (I-c)^{-1}(I-c^{s-1})\boldsymbol{W}$ and $c = (\alpha + \beta)$.

Next, the DCC-HEAVY model s-step ahead forecast $\mu_{t+s|t}$ and $\mathbf{\Phi}_{t+s|t}$ can be derived from eq.(19) and eq.(20) by setting $A=\begin{bmatrix} 0 & A_h \\ 0 & A_m \end{bmatrix}$, $B=\begin{bmatrix} B_h & \\ & B_m \end{bmatrix}$, $\alpha=\begin{bmatrix} & \alpha_r \\ 0 & \alpha_p \end{bmatrix}$, $\beta=\begin{bmatrix} \beta_r & \\ & \beta_p \end{bmatrix}$.

After obtaining $\mu_{t+s|t}$ and $\mathbf{\Phi}_{t+s|t}$, the s-step ahead forecast of condition variance $E_t[h_{t+s}]$ is the first k elements of $\mu_{t+s|t}$ and the s-step forecast of condition correlation $E_t[R_{t+s}]$ is the first k elements of $\mathbf{\Phi}_{t+s|t}$. Then, the s-step ahead forecast of $E_t[H_{t+s}]$ is given by

$$E_t[H_{t+s}] = E_t[di\alpha g(h_{t+s}^{1/2})]E_t[R_{t+s}]E_t[di\alpha g(h_{t+s}^{1/2})]. \tag{21}$$

For out-of-sample forecast evaluation, we use two loss functions. The first one a quasi-likelihood (QLIK) loss function (e.g., Noureldin et al., 2012) of the form

$$L_{t,s}(\Sigma_{t+s}, H^a_{t+s|t}) = |\Sigma_{t+s}| + tr((H^a_{t+s|t})^{-1}\Sigma_{t+s}) - \log|(H^a_{t+s|t})^{-1} - k \tag{22}$$

where Σ_{t+s} is the actual (unobserved) covariance matrix and $H^a_{t+s|t}$ denotes its s-step forecast using model a conditional on time t information.

The second loss function is the root-mean-squared error (RMSE) based on the Frobenius norm (Chiriac and Voev, 2011 and Golosnoy et al. , 2012) of the forecast error, which is defined by

$$FN_s = \frac{1}{s}\sum_t ||\Sigma_{t+s} - H^a_{t+s|t}|| = \frac{1}{s}\sum_t \left[\sum_{i,j}(\sigma_{ij,t+s} - h_{ij,t+s})^2\right]^{1/2}. \tag{23}$$

Since Σ_{t+s} is unobservable, our analysis will be based on some proxy denoted by Σ_{t+s}, which we take to be the realized covariance matrix, RC_{t+s}. The loss function evaluates the s-step predicted density from model a using the proxy $\hat{\Sigma}_{t+s}$ as data, and it provides a consistent ranking of volatility models in the sense of Patton (2011) and Patton and Sheppard (2009) as it is robust to noise in the proxy $\hat{\Sigma}_{t+s}$; see also Laurent et al. (2013).

3.3 DCC-HEAVY model extension

It has long been recognized that financial markets react differently to positive and negative news. The asymmetric effect is now commonly used to refer to any volatility model, univariate and multivariate alike, in which the (co)variances respond asymmetrically to positive and negative shocks. The DCC-HEAVY model can be extended by incorporating the asymmetric effect to both the variance and correlation equation. In the variance equation, the asymmetric effect implies that volatility tends to increase more following negative return shocks than after equally-sized positive shocks. In the correlation equation, the asymmetric effect implies that the correlation between stock returns tends to increase when the market turns down. The extended model is called ADCC-HEAVY model.

Defing D_t=$di\alpha g[d_t] = di\alpha g(d^h_{1t}, d^h_{2t}, ..., d^h_{kt})$, where $d^h_{jt} = 1$ if $r_{jt} < 0$ and $d^h_{jt} = 0$, if $r_{it} \succeq 0$ for $i = 1, .., k$, and define $D^r_t = D_t \ D'_t$, the dynamics of the conditional covariance matrix in the asymmetric ADCC-HEAVY model are

$$
\begin{aligned}
H_t &= di\alpha g[h_t^{1/2}] R_t di\alpha g[h_t^{1/2}] \qquad (24)\\
h_t &= \omega_h + A_h v_{t-1} + B_h h_{t-1} + \gamma_h D_{t-1} v_{t-1}\\
R_t &= \tilde{R} + \alpha_r RL_{t-1} + \beta_r R_{t-1} + \gamma_r D^r_{t-1} RL_{t-1}
\end{aligned}
$$

where $\tilde{R} = (1 - \beta_r) \ \bar{R} - (\alpha_r + \gamma_r \bar{D}^r)\bar{P}$ is a $k \times k$ matrix and $\bar{D}^r$ equals the unconditional mean D^r_t . If asymmetric effects are significant, then γ_h and γ_r should be positive.

We can also add asymmetric effects in the realized covariance equations. The dynamics of the realized covariance matrix in the asymmetric form are

$$\begin{aligned}
M_t &= di\alpha g[m_t^{1/2}]P_t di\alpha g[m_t^{1/2}]. \\
m_t &= \omega_m + A_m v_{t-1} + B_m m_{t-1} + \gamma_m D_{t-1} v_{t-1} \\
P_t &= \tilde{P} + \alpha_p RL_t + \beta_p P_{t-1} + \gamma_r D_{t-1}^r RL_{t-1}
\end{aligned}$$

Furthermore, the heterogeneous autoregressive (HAR) model of Corsi (2009) has arguably emerged as the most widely used univariate realized volatility-based forecasting model. The model was extended to a multivariate setting by Chiriac and Voev (2010) and Oh and Patton (2016). We add the HAR model structure to the realized covariance equations [7]:

$$\begin{aligned}
M_t &= di\alpha g[m_t^{1/2}]P_t di\alpha g[m_t^{1/2}]. \\
m_t &= \omega_m + A_m v_{t-1} + B_m m_{t-1} + \gamma_m D_{t-1} v_{t-1} + \alpha_m^w v_{t-1}^w + \alpha_m^m v_{t-1}^m \\
P_t &= \tilde{P} + \alpha_p RL_t + \beta_p P_{t-1} + \gamma_p D_{t-1}^r RL_{t-1} + \alpha_p^w RL_{t-1}^w + \alpha_p^m RL_{t-1}^m
\end{aligned} \tag{25}$$

where $v_{t-1}^w = \frac{1}{5}\sum_{j=1}^{5} v_{t-j}$ and $v_{t-1}^m = \frac{1}{22}\sum_{j=1}^{22} v_{t-j}$ and $RL_{t-1}^w = \frac{1}{5}\sum_{j=1}^{5} RL_{t-j}$ and $RL_{t-1}^m = \frac{1}{22}\sum_{j=1}^{22} RL_{t-j}$, so $\tilde{P} = (1 - \alpha_p - \beta_p - \gamma_r \bar{D}^r - \alpha_p^w - \alpha_p^m)\bar{P}$.

The out-of-sample forecasts of ADCC-HEAVY model can be derived recursively. The formulas are provided in the additional appendix.

4 Estimation and Inference

DCC-HEAVY model is parameterized with a finite-dimensional $(\delta \times 1)$ parameter vector $\theta \in \Theta \subset \mathbb{R}^\delta$. Decompose $\theta = (\theta'_H, \theta'_M)$ where the $(\delta_H \times 1)$ vector θ_H and the $(\delta_M \times 1)$ vector θ_M denote the parameters vector in the conditional covariance and realized covariance equations, respectively. θ_H and θ_M could be estimated separately, as they are variation free in the sense of Engle et al. (1983). Then θ_H and θ_M in DCC-HEAVY model can be estimated by two-step estimation as proposed by Engle (2002).

[7] Note we also test the HAR structure in the conditional covariance equations, but the effects are insignificant, so we only add HAR structure in the realized covariance equations.

4.1 Estimation of θ_H in the condition covariance equations

Now let's write the return vector r_t by the relation

$$
\begin{aligned}
r_t &= u_t \odot h_t^{1/2} \;,\; u_t|\mathcal{F}_{t-1} \sim N(0, R_t) && (26)\\
H_t &= di\alpha g[h_t^{1/2}] R_t di\alpha g[h_t^{1/2}] && (27)
\end{aligned}
$$

A natural choice of density for the innovation of return is the multivariate Gaussian distribution, the log-likelihood for returns equation can be written as

$$
\begin{aligned}
L_H(\theta_H; r_t) &= -\frac{1}{2}\sum_{t=1}^{T}(k\log(2\pi) + \log(|H_t|) + r_t' H_t^{-1} r_t) && (28)\\
&= -\frac{1}{2}\sum_{t=1}^{T}\Big(k\log(2\pi) + 2\log(|di\alpha g(h_t^{1/2})|) + \log(|R_t|) + u_t' R_t^{-1} u_t\Big)
\end{aligned}
$$

The DCC type model can also be estimated by two-step estimation as proposed by Engle (2002). The parameter space θ_H is split into θ_{H1} for the parameters in the variance equation and θ_{H2} for the parameters in the correlation equation. We denote by QL_{H1} the likelihood where R_t in (28) is replaced by the identity matrix

$$
L_{H1}(\theta_{H1}; r_t) = -\frac{1}{2}\sum_{t=1}^{T}\Big(k\log(2\pi) + 2\log(|di\alpha g(h_t^{1/2})|) + u_t' u_t\Big) \tag{29}
$$

We denote by QL_{H2} the likelihood given θ_{H1} where we have concentrate out h_t:

$$
L_{H2}(\theta_{H2}; r_t, \theta_{H1}) = -\frac{1}{2}\sum_{t=1}^{T}\big(k\log(2\pi) + \log(|R_t|) + u_t' R_t^{-1} u_t\big) \tag{30}
$$

The DCC-HEAVY model is a two-step estimator of conditional variance and correlation. Two-step estimation is simple, and an inefficient but consistent estimator of the parameter θ_H can be found.

Engle (2002) show that this two steps procedure produces consistent maximum likelihood parameter estimates. The two-step approach makes the estimation much simple, and hence the DCC type of model is applicable to large conditional variance-covariance matrix estimation, without imposing a scalars restriction on the parameters.

4.2 Estimation of θ_M in the realized covariance equations

Since realized covariance is a symmetric positive define matrix, a natural choice of density for the innovation in the realized covariance equation is the Wishart distribution.

We assume RC_t follows a $n-$dimensional central Wishart distribution and denote this assumption by

$$RC_t|\mathcal{F}_{t-1} \sim W_k(\nu, M_t/\nu)$$

where $\nu(> k-1)$ is the degree of freedom. From the properties of the Wishart distribution - see e.g. Anderson (1984) and Golosnoy, Gribisch, and Liesenfeld (2012)- it follows that

$$E(RC_t|\mathcal{F}_{t-1}) := M_t$$

Using the expression of a Wishart density function, and of M_t in (5), we obtain the loglikelihood contribution

$$\begin{aligned} L_M(\theta_M; RC_t) &= \sum_{t=1}^{T} \Big\{ C_M - \frac{v}{2} \log(|M_t|) - \frac{v}{2} tr(M_t^{-1} RC_t) \Big\} \quad (31) \\ &= C_M + \sum_{t=1}^{T} \Big\{ -\frac{v}{2} \log(|V_t P_t V_t|) - \frac{v}{2} tr((V_t P_t V_t)^{-1} RC_t) \Big\} \end{aligned}$$

where $C_M = \frac{vk}{2} \log \frac{v}{2} + \frac{(v-k-1)}{2} \log |RC_t| - \sum_{i=1}^{k} \log \Gamma[v+1-i)/2]$ which are constants with respect to θ_M.

The DCC-HEAVY model for realized covariance matrix can also be estimated in two steps. The parameter space θ_M is split into θ_{M1} for the parameters in the realized volatility model and θ_{M2} for the parameters in the realized correlation model. We denote by L_{M1} the likelihood where P_t in (31) is replaced by the identity matrix.

$$L_{M1}(\theta_{M1}; RC_t) = -\sum_{t=1}^{T} \Big\{ v \log(|V_t|) + \frac{v}{2} tr(V_t^{-1} RC_t V_t^{-1}) \Big\} \quad (32)$$

We denote by L_{M2} the likelihood given θ_{M1} where we have concentrate out v_t:

$$L_{M2}(\theta_{M2}; RC_t, \theta_{M1}) = -\frac{v}{2} \sum_{t=1}^{T} \big\{ \log(|P_t|) + tr((P_t^{-1} - I_k) V_t^{-1} RC_t V_t^{-1}) \big\} \quad (33)$$

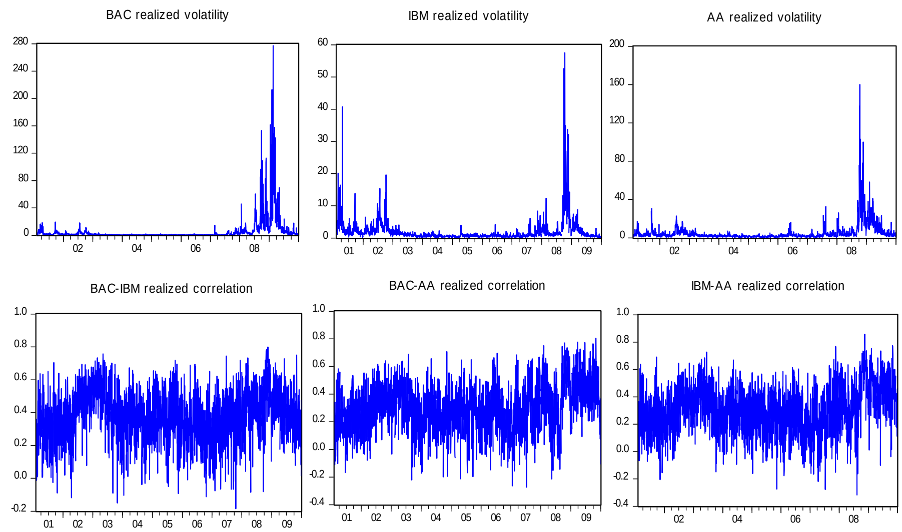

Figure 1: Figure 1. Annualized realized volatility and realized correlation.

Two-step estimation is simple, and an inefficient but consistent estimator of the parameter θ_M can be found.

5 Empirical Application

5.1 Data

We use the same data as Noureldin et al. (2012), which is available to download from JAE data archive.

They use high-frequency data from 10 most liquid stocks in the Dow Jones Industrial Average (DJIA) index. These are: Alcoa (AA), American Express (AXP), Bank of America (BAC), Coca Cola (KO), Du Pont (DD), General Electric (GE), International Business Machines (IBM), JP Morgan (JPM), Microsoft (MSFT), and Exxon Mobil (XOM). The sample period is 1 February 2001 to 31 December 2009 with a total of 2242 trading days, and the data source is the TAQ database.

The main focus of the empirical application is on modelling and forecasting the conditional covariance matrix of all 10 stocks using the DCC-HEAVY model. We slso estimate 5 stocks, 4 stocks, 3 stocks and 2 stocks combinations for robustness check.

Figure 1 contains the annualized realized volatility of BAC, IBM and AA, and their realized correlation over the full sample. The sharp increase in volatility in 2008–2009 is associated with the turmoil in financial markets during the recent financial crisis. The increase in volatility is much more pronounced especially after the collapse of Lehman Brothers in mid September 2008. The realized correlation is very persistent. And the realized correlation seems to have been relatively high during the crisis.

5.2 Empirical results

In Table I, we present the HEAVY, GARCH and GARCHX model estimates. This is the first step parameter estimates in the variance equation. The estimate of β_r in HEAVY model is smaller than that in GARCH model, implying that the elements of volatility in HEAVY model will be smooth, although less smooth than the corresponding estimates from the GARCH model. Compared to the nesting GARCH-X model, there is no loss of fit when moving to HEAVY-R since the coefficient on α_{rr} in GARCH-X model is not statistically significant. This is not the case when moving from GARCH-X to GARCH, which suggests that conditional variance is effectively driven by lagged realized variance.

In Table 2, we present the DCC-HEAVY, DCC-GARCH and DCCX-GARCH model estimates. This is second step parameter estimates in the conditional correlation equation. These estimates are largely in line with those from the univariate HEAVY model in Table 1 and Shephard and Sheppard (2010). The estimate of β_r in DCC-HEAVY model is smaller than that in DCC-GARCH model, implying that the elements of correlation in DCC-HEAVY model will be smooth, although less smooth than the corresponding estimates from the DCC-GARCH model. Compared to the nesting DCCX-GARCH model, there is no loss of fit when moving to DCC-HEAVY since the coefficient on α_{rr} in DCCX-GARCH model is not statistically significant. This is not the case when moving from DCCX-GARCH to DCC-GARCH, which suggests that conditional correlation is effectively driven by lagged realized correlation.

The estimates also suggest that the DCC-HEAVY model's half-life (of a deviation of the one-step forecast of conditional variance or correlation from its long run) is substantially shorter than that of the DCC-GARCH model, suggesting that the former's forecast responds faster to abrupt changes in the level of volatility or correlation.

Table 1: Step 1 parameter estimates in the variance equation

	GARCH		GARCHX			HEAVY-R		HEAVY-RM	
	α_{rr}	β_r	α_{rr}	β_r	α_{rR}	β_r	α_{rR}	α_m	β_m
BAC	0.067	0.927	0.003	0.644	0.604	0.641	0.614	0.474	0.526
	(2.53)	(32.70)	(0.27)	(7.11)	(3.20)	(7.59)	(3.73)	(9.86)	(10.96)
JPM	0.086	0.914	0.007	0.767	0.347	0.763	0.363	0.501	0.499
	(4.89)	(52.30)	(0.47)	(10.68)	(2.82)	(10.83)	(3.19)	(11.30)	(11.27)
IBM	0.106	0.879	0.034	0.513	0.699	0.499	0.782	0.352	0.634
	(1.09)	(8.02)	(1.40)	(3.37)	(2.67)	(3.11)	(2.95)	(9.20)	(16.10)
DD	0.061	0.931	0.005	0.655	0.453	0.652	0.463	0.366	0.618
	(2.98)	(41.60)	(0.22)	(7.99)	(3.34)	(8.65)	(4.22)	(10.71)	(17.39)
XOM	0.082	0.895	0.008	0.695	0.339	0.691	0.354	0.378	0.603
	(5.53)	(47.60)	(0.45)	(10.96)	(4.05)	(11.31)	(4.87)	(11.35)	(17.50)
AA	0.045	0.948	0.000	0.868	0.197	0.868	0.197	0.314	0.669
	(4.22)	(72.92)	(.)	(4.91)	(0.76)	(4.91)	(0.76)	(9.47)	(19.41)
AXP	0.095	0.905	0.001	0.727	0.442	0.726	0.444	0.397	0.603
	(6.82)	(64.76)	(0.13)	(13.30)	(4.57)	(13.66)	(4.81)	(10.65)	(16.20)
DD	0.061	0.931	0.005	0.655	0.453	0.652	0.463	0.366	0.618
	(2.98)	(41.60)	(0.22)	(7.99)	(3.34)	(8.65)	(4.22)	(10.71)	(17.39)
GE	0.054	0.945	0.011	0.478	0.837	0.509	0.799	0.369	0.631
	(3.29)	(57.56)	(0.12)	(1.16)	(1.31)	(23.81)	(.)	(11.38)	(19.45)
KO	0.105	0.888	0.057	0.556	0.515	0.491	0.659	0.356	0.631
	(1.70)	(14.20)	(1.69)	(4.78)	(2.83)	(4.47)	(4.36)	(10.61)	(18.58)

Note: robust t statistics are reported in the bracket.

The log-likelihood and its decomposition into variance and correlation equation are reported in Penal B of Table 2. It indicates an improvement in fit of the DCC-HEAVY model compared to the DCC-GARCH model. The decomposition suggests that the DCC-HEAVY improves on DCC-GARCH model in both the variance and correlation equation. However, most of the in sample gains are coming from the variance equation. The overall improvement is substantial.

Table 3 gives the results of the out of sample forecasting comparison. We estimate the model using a rolling window of 1486 observations. The size of the rolling window is chosen such that our forecasts start on 3 January 2007. The forecast period 3 January 2007 to 31 December 2009. We use the parameter estimates to obtain forecasts of $H_{t+s|t}$ at horizons $s = 1, 5, 22$ days using eq.(21). Table 3 reports the ratio of the losses for the DCC-HEAVY model relative to the losses of the DCC-GARCH model at the different forecast horizon.

The results show that DCC-HEAVY outperforms DCC-GARCH, especially at short forecast horizons. This is true for the whole covariance matrix forecast as well as its decomposition into variance and correlation components, which

Table 2: Step 2 parameter estimates in the correlation equation

DCC-GARCH		DCCX-GARCH			DCC-HEAVY		HEAVY-RC	
α_{rr}	β_r	α_{rr}	β_r	α_{rR}	β_r	α_{rR}	α_m	β_m
0.010	0.974	0.012	0.795	0.106	0.761	0.131	0.077	0.908
(4.58)	(137.1)	(1.33)	(20.3)	(5.04)	(17.16)	(6.24)	(22.45)	(195.8)

Panel B: Log-likelihood decomposition (DCC-HEAVY vs DCC-GARCH)

	DCC-GARCH	DCC-HEAVY	DCC-HEAVY Gains
Variance	-4316	-4249	67
Correlation	-2714	-2705	9
Joint distribution	-7030	-6955	76

Note: robust t statistics are reported in the bracket. HEAVY-RC is the parameter estimates of realized correlation equation.

provides further insight into the source of forecast gains. Based on RMSE as loss function, it can be seen that the overall forecast of the covaraince matrix gain is relatively large. The DCC-HEAVY model reduces the forecast error of the covariance matrix by 30% compared with the DCC-GARCH model. Based on QLIK as loss function, the overall forecast gain is still substatial. The only exeption is the 1 month (22 days) ahead forecasts, where DCC-GARCH model outperfoms DCC-HEAVY model by 4.9%. Overall, our empirical results show the superior forecasting ability of DCC-HEAVY model. Consistent with the in sample log-likelihood statistics, most of the forecasting gains are coming from variance equations. But it does suggest that the realized measure provides valuable information for forecasting the conditional correlation.

We also estimate the ADCC-HEAVY model and forecasting the conditional covariance matrix by using ADCC-HEAVY model. For sake of space, we only report the out of sample forecast results in the last two colums of Table 3. The two step estimation results for ADCC-HEAVY model can be found in the additional appendix. The results in Table 3 show that the ADCC-HEAVY further improve the out-os-sample forecasts accuracy. Based on RMSE as loss function, it can be seen that the overall forecast of covariance matrix gain is large. The DCC-HEAVY model reduces the forecast error by 40% compared with the DCC-GARCH model. Again, most of the forecasting improvements are coming from variance equations.

Table 3: Ratio of the losses(DCC-HEAVY and ADCC-HEAVY vs DCC-GARCH)

		DCC-HEAVY		ADCC-HEAVY	
		RMSE	QLIK	RMSE	QLIK
Covariance	s=1	0.7118	0.7303	0.6391	0.7841
	s=5	0.7453	0.8135	0.6092	0.8143
	s=22	0.7616	1.0488	0.6751	0.9703
Variance	s=1	0.8720	0.6667	0.8336	0.6534
	s=5	0.8566	0.7648	0.7760	0.7296
	s=22	0.8828	0.9713	0.8399	0.9360
Correlation	s=1	0.9231	0.9254	0.9355	0.9902
	s=5	0.9547	0.9854	0.9645	0.9810
	s=22	0.9877	1.0400	1.0084	0.9825

Note: Ratio of the losses for the DCC-HEAVY and ADCC-HEAVY models relative to the losses of the DCC-GARCH model

5.3 Robustness

For robustness, we estimate the DCC-HEAVY model including only 5 stocks. We divide the ten stocks into two paris, each pair includes 5 stocks. The first step variance equation estimation is exactly the same as Table 1. We report the second step correlation equation estimation in Table 4 and 5. The out of sample comparison results is reported in Table 6.

Table 4: Step 2 correlation estimation results (5 stocks pair 1)

DCC-GARCH		DCCX-GARCH			DCC-HEAVY		RC-HEAVY	
0.009	0.981	0.006	0.686	0.171	0.641	0.195	0.102	0.879
(3.64)	(140.9)	(0.98)	(2.23)	(1.12)	(2.24)	(1.49)	(16.29)	(104.7)

Panel B: Log-likelihood decomposition (DCC-HEAVY vs DCC-GARCH)

	DCC-GARCH	DCC-HEAVY	DCC-HEAVY gains
Variance	-4339	-4287	53
Correlation	-2860	-2854	6
Joint distribution	-7199	-7140	59

The results are consistent with 10 stocks estimation results. It can be seen that the conditional correlation is mainly driven by the lagged realized correlation. The DCC-HEAVY model outperforms DCC-GARCH model both in and out of the sample. However, most of the gains are coming from the variance equations.

We also estimate 4 stocks, 3 stocks and 2 stocks conditional correlation equa-

Table 5: Step 2 correlation estimation results (5 stocks pair 2)

DCC-GARCH		DCCX-GARCH			DCC-HEAVY		RC-HEAVY	
0.027	0.927	0.026	0.722	0.156	0.653	0.218	0.114	0.871
(2.62)	(25.28)	(2.58)	(13.22)	(3.82)	(11.79)	(6.46)	(17.66)	(108.9)

Panel B: Log-likelihood decomposition (DCC-HEAVY vs DCC-GARCH)

	DCC-GARCH	DCC-HEAVY	DCC-HEAVY gains
Variance	-4293	-4212	81
Correlation	-2825	-2815	10
Joint distribution	-7118	-7028	91

Table 6: Ratio of the losses (DCC-HEAVY vs DCC-GARCH)

		5 assets pair 1		5 assets pair 2	
		RMSE	QLIK	RMSE	QLIK
Covariance	s=1	0.8103	0.6929	0.6370	0.7760
	s=5	0.8290	0.7884	0.6810	0.8070
	s=22	0.7940	0.9642	0.7260	1.1150
Variance	s=1	0.8406	0.6627	0.9030	0.6710
	s=5	0.8699	0.7666	0.8430	0.7630
	s=22	0.8802	0.9342	0.8850	1.0080
Correlation	s=1	0.9375	0.9492	0.9070	1.7410
	s=5	0.9636	0.9530	0.9360	1.1600
	s=22	0.9853	0.9580	0.9780	1.0570

Note: this table reports the ratio of the losses for the DCC-HEAVY models relative to the losses of DCC-GARCH models.

tions and compare the forecasting performance with DCC-GARCH model. The main conclusion always holds. The additional appendix provides the detailed results.

6 Conclusion

This paper proposes a new class of multivariate volatility model utilising high-frequency data. We call this model DCC-HEAVY as key ingredients are Engle (2002) DCC model and Shephard and Sheppard (2010) HEAVY model. In the DCC-HEAVY model, the dynamics of conditional variances are driven by the lagged realized variances, while the dynamics of conditional correlations are driven by the lagged realized correlations. The DCC-HEAVY model removes the well know asymptotic bias (Aielli 2013) in DCC-GARCH estimation, has more desirable asymptotic properties. We also derive a Quasi-maximum likelihood estimation and provide closed-form formulas for multi-step forecasts. Empirical results suggest that the DCC-HEAVY model outperforms the DCC-GARCH model in and out-of-sample.

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