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Abstract

A representative pensioner is considered for the evaluation of some of the cost factors for the career-average-revalued-earnings (CARE) defined benefit scheme of USS (the Universities Superannuation Scheme). Since the promised benefit increases with inflation, the return on the pension portfolio is required to exceed the rate of inflation in order for the scheme to be fully funded. Therefore, given current low interest rates, the de-risking of replacing equities with bonds in the portfolio significantly increases the risk of under-funding and hence is prohibitively expensive. On the other hand, the risk of holding equities in the portfolio can be effectively mitigated through the principle of time diversification, thereby resulting in not only a high probability of a fully-funded scheme, but also possible lower contributions from both scheme sponsors and members in the future. Moreover, it is shown that a zero or negative real interest rate provides the condition for allocating all funds into equities if minimising the probability of underfunding is the sole objective. Finally, the paper finds that the promised benefit will be cheaper to fund if the pensioner has (i) made more years of contribution; (ii) has become a deferred member; (iii) has a slower wage growth; and (iv) has made the contribution earlier. The implication is that the current CARE scheme is cheaper and less risky than the final-salary scheme.

Keywords: pension, defined benefit, time diversification, de-risking, USS

First draft: 31 October 2018

NOTE: My apology as the paper is completed in a rush and hence not well written, but hopefully its early availability would help discussion on issues surrounding USS

1. Introduction

In July 2017, the USS defined benefit scheme reported the biggest deficit of any British pension fund.[1] This prompted the USS Joint Negotiating Committee to propose to close the defined benefit scheme from 1 April 2019 onward. The University and College Union (UCU) objected to the proposal but was overruled. Consequently, the largest ever industrial action seen in UK universities began on 22 Feb 2018. Unlike other industrial actions that mostly only aim to secure a more favourable settlement for university staff, the pension dispute also highlights the controversy surrounding USS valuation.[2] As a result, when the strike ended, the Joint Expert Panel (JEP) was formed to reconsider how valuations should be undertaken. The report published on 13 Sep 2018 by JEP points out the 'excessive complexity' of the USS valuation and that the large volume of information cannot be used as a 'substitute for good quality information' for scheme members to understand the valuation.[3] The JEP report is followed by claims of various academics that de-risking, i.e. replacing equities with bonds in the pension portfolio, is not only unnecessary but actually the cause of USS's funding predicament.[4] USS immediately responded by pointing out that while the claim is 'not wrong in isolation', the reported deficit is a reflection of the need to protect the scheme from downside risk as well as if the interest rates' expected rise does not materialise in the future.[5] So, the dispute continues.

In response to the JEP report, this paper aims to establish some of the key cost factors for the career-average-revalued-earnings (CARE) scheme, so that to facilitate communication and understanding between USS and scheme members. The asset as a result of the contributions made by a representative pensioner as well as the associated accrued benefit is considered. Since the promised pension increases annually at rate of inflation, the cost is expressed in terms of real rate of return on the pension portfolio. Given the current contribution rate of 22.2%, in the case of static inflation and expected returns, the return on the pension portfolio is required to exceed inflation by around 1.2% in order to achieve a funding ratio of 100% for USS CARE. If uncertainty is introduced, the required rate of real return is less than 2.2%, with prudence. Assuming real returns on equities and bonds as 5% and 0% respectively, a balanced portfolio of half equities half bonds would provide sufficient return for the scheme to be fully funded. However, if de-risking were to take place resulting in a pension portfolio that is predominately bonds with, the contributions by both universities and scheme members need to rise significantly in order to ensure a fully funded scheme.

The other objective of the paper is to provide a theoretical model to assess the risk of equities in the pension portfolio. The return on the asset of a representative pensioner is modelled as a stochastic compound return over a n-year time horizon. The compound return

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[1] The Economist, 3 August 2017, Universities' main pension pot faces the biggest deficit of any British fund.
[2] For example, Wong (2018) shows that despite USS's complex valuation process, one single interest rate can explain up to 99.3% of USS's liability in the past seven years; moreover, the deficit turns into surplus if an appropriate discount rate is used.
[3] See page 9 and 10 of the JEP report, available from: https://www.ucu.org.uk/uss-jep-report
[4] See article #USSbriefs58 and #USSbriefs59 available from: https://medium.com/ussbriefs
[5] See https://www.uss.co.uk/how-uss-is-run/valuation/2017-valuation-updates/claims-of-a-large-and-demonstable-error

can in turn be transformed into sum of n stochastic returns, which are assumed to be independent and identically distributed (IID). It is well known that while the average of n IID random returns remains unchanged, the risk as measured in standard deviation is reduced by a factor of $\sqrt{n}$ – referred to as time diversification in this paper. Defining the risk of a defined benefit scheme as the probability of its asset less than its liability n years later, it is possible to calculate the risk in terms of the probability that the return on the pension portfolio fails to meet the inflation cost. Therefore, based on the principle of time diversification, the risk of equities in a pension portfolio can be effectively mitigated by the long holding time horizon.

The concept of time diversification discussed above does not necessary imply that more funds should be allocated to equities if a portfolio is held for a long horizon. Conditions for holding more equities in a pension portfolio leading to lower risk of underfunding are derived. An important finding is that a zero or negative real interest rate provides the required condition for allocating most if not all funds into equities in order to maximise the chance of a fully funded scheme.

The results presented in this paper would also help stakeholders of USS to understand why the current CARE scheme in general is less risky than the final-salary scheme. Educated guesses are made which suggest that the USS is fully funded and is perfectly sustainable provided the planned de-risking does not take place.

Finally, it is remarked that while the examples discussed in this paper refer specifically to USS, the findings are applicable to other defined benefit schemes, especially the result on the time diversification of equity risk. More importantly, the finding of this paper also raises a serious question as to whether the previous closure of many defined benefit schemes is actually justified.

The paper is organised as follows. Section 2 presents the model of a representative pensioner. The required rate of real returns are provided for the case of static inflation and expected returns. Section 3 provides the stochastic model to show how the risk of holding equities in a pension portfolio can be diversified through long time horizon. Section 4 uses simulation to investigate the CARE scheme of representative pensioners and provides evidence to show that USS is fully funded. Finally, section 5 concludes.

2. A representative pensioner

Consider a pensioner who has made N years of contribution to the CARE scheme of USS and is now at the beginning of retirement. Let A_t and $P_{y,t}$ be respectively the asset accumulated and the annual pension accrued as a result of the contribution made in year t. Then the total asset accumulated at retirement is $A = \sum_{t=1}^{N} A_t$ whereas the associated accrued annual pension is $P_y = \sum_{t=1}^{N} P_{y,t}$. Note that the other part of the benefit is the accrued lump sum $P_{ls} = 3P_y$, which is payable at the beginning of retirement. As it can be seen later in this section, A_t and $P_{y,t}$ are the 'basic units' that help us to understand the various cost factors of a defined benefit scheme.

2.1 Asset and liability due to contribution in year t

Let £W_t be the wage in year t and $r_1, \dots, r_n$ be the subsequent returns on the pension portfolio. For simplicity, we assume there is no cap on the level of wage for the defined benefit to be applicable.[6] It is noted that $n = N - t$ is the number of years before retirement after making contribution in year t. If a is the rate of combined contribution from both university and the scheme member, then the asset grows in n years' time to

$$A_t = aW_t \prod_{s=1}^{n} (1 + r_s). \tag{1}$$

Let $\pi_1, \dots, \pi_n$ be the corresponding rates of inflation and b be the accrual rate. Then the associated accrued annual pension is

$$P_{y,t} = bW_t \prod_{s=1}^{n} (1 + \pi_s). \tag{2}$$

Note that the associated accrued lump sum is

$$P_{ls,t} = 3P_{y,t}. \tag{3}$$

Let i_g and r_g be the geometric average of the rates of inflation and investment returns respectively. Then the funds available for each Sterling pound of accrued pension at the beginning of retirement is

$$F_t = \frac{A_t - P_{ls,t}}{P_{y,t}} = \frac{aW_t \prod_{s=1}^{n}(1 + r_s)}{bW_t \prod_{s=1}^{n}(1 + \pi_s)} - 3 = \frac{a}{b}\left(\frac{1 + r_g}{1 + \pi_g}\right)^n - 3. \tag{4}$$

Note that for USS, $a = 0.222$ and $b = 1/75$.[7]

Now, let e be the life expectancy at retirement. Then for each Sterling pound of $P_{y,t}$, the sum of future annual pensions to be paid out is

$$\sum_{s=1}^{e} \prod_{\tau=1}^{s} (1 + \pi_{s-1}), \tag{5}$$

where $\pi_0 = 0$ and $\pi_1, \dots, \pi_{e-1}$ here refer to the rates of inflation during the period of retirement. Given a discount rate r, there exists an 'average' rate of inflation $\tilde{\pi}$ for which the present value of the above future annual payments can be obtained as below

[6] The cap on USS's defined benefit scheme is currently set at £55,550. As can be seen later, the introduction of cap produces little difference in the results obtained.

[7] Total contributions from both employer and scheme members are 26%, out of which 22.2% is for the defined benefit part of the USS; see First Actuarial (2017).

$$L = \frac{1}{1+r}\sum_{s=1}^{e}\prod_{\tau=1}^{s}\left(\frac{1+\pi_{s-1}}{1+r}\right) = \frac{1}{1+r}\sum_{\tau=1}^{e}\left(\frac{1+\tilde{\pi}}{1+r}\right)^{\tau-1} = \frac{1-(1-\delta)^e}{(1+r)\delta} \tag{6}$$

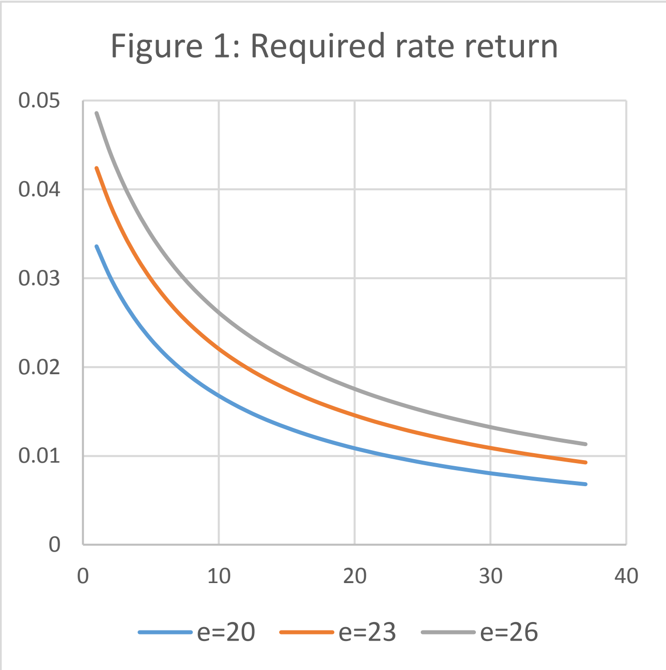
where $\delta = (r-\tilde{\pi})/(1+r)$, assuming that $r > \tilde{\pi}$.

2.2 The required return

It is useful to determine the required rate of return on the pension portfolio that equates the available funds F_t with the liability L. In order to facilitate our understanding of the relevant cost factors, consider the static case in which $r_g = E(r_s) = r$ and $\pi_g = \tilde{\pi} = E(\pi_s) = \pi$. Equating (4) to (6) gives

$$\frac{a}{b}\left(\frac{1+r}{1+\pi}\right)^n - 3 = \frac{1-(1-\delta)^e}{(1+r)\delta}. \tag{7}$$

Setting the rate of inflation $\pi = 2\%$, Figure 1 below depicts the required rate of real return $(r-\pi)$ for various value of n, the number of years after making the contribution, and with life expectancy at retirement e = 20, 23 and 26.



It can be seen that higher e and lower n would require a higher level of return on the available funds or assets to fund the future pensions. The intuition for the inverse relationship between r and n is that with the life expectancy e fixed, the longer the number of years for the asset to grow above the inflation, the lower is the required rate of return for the available fund to match the liability.

The relationship between r and n has the following implications. First, members who join the scheme earlier and hence stay longer in the scheme cost less than those who join the scheme later in life, others being equal. Second, the benefits accrued toward the end of one's working life are more expensive to fund, as compared to the benefits accrued at the beginning. This explains the cap on the pensionable wage; it will reduce the cost as wages tend to go higher as one ages. Last but not least, for deferred members who leave the scheme before retirement, the cost is cheaper as their benefits are accrued earlier in their working life.

Perhaps more importantly, the above analysis sheds light on the effect of de-risking currently undertaken by USS. To help readers to appreciate the impact of replacing equities with bonds consider the following equation which is an approximate version of (7).[8]

$$\text{(Funds) } \frac{a}{b}\{1+n(r-\pi)\}-3=e-0.5e^2(r-\pi) \text{ (Liability)} \tag{8}$$

Equation (8) highlights the importance of having the return on asset sufficiently larger than the rate of inflation. A lower real return on the asset not only reduces the available fund but also increases the liability. Given the current negative real interest rates, replacing equities with bonds will significantly lower r, resulting in large deficit unless the contribution rate a is raised and/or the accrual rate b is lowered.

While holding more equities in the pension portfolio would solve the funding problem, critiques point out that such portfolio is too risky. The next two sections of this paper will show theoretically and through simulation that the risk can be effectively mitigated based on the principle of time diversification. The remainder of this section shows that the results presented here for the asset and liability of a single-year contribution can also be extended to the case of a representative pensioner where all N years of contribution are considered.

2.3 Asset and liability after N years of contribution

Now consider the case in which the pensioner has made N years of contribution before retirement.

Let $W_1, \dots, W_N$ be the wages during the N years of contribution. Let ω_t be the rate of wage rise at the end of year t such that $W_t = W_1 \prod_{s=1}^{t-1}(1+\omega_s)$ for $2 \le t \le N$. The accumulated asset after N years of contribution can be expressed as

$$A = a\sum_{t=1}^{N} W_t \prod_{s=t+1}^{N}(1+r_s) = aW_1 \sum_{t=1}^{N}\left\{\prod_{s=1}^{t-1}(1+\omega_t)\prod_{s=t+1}^{N}(1+r_s)\right\} \tag{9}$$

Note that for each t, the summand is a product of $t-1$ one-plus-wage-rises and $N-t$ one-plus-portfolio-returns. There exists r_ω such that

$$A = NaW(1+r_\omega)^{N-1}. \tag{10}$$

Similarly for the annual pension,

$$P_y = b\sum_{t=1}^{N} W_t \prod_{s=t+1}^{N}(1+\pi_s) = bW_0 \sum_{t=1}^{N}\left\{\prod_{s=1}^{t-1}(1+\omega_t)\prod_{s=t+1}^{N}(1+\pi_s)\right\}, \tag{11}$$

there exists π_ω such that

$$P_y = NbW_0(1+\pi_\omega)^{N-1}. \tag{12}$$

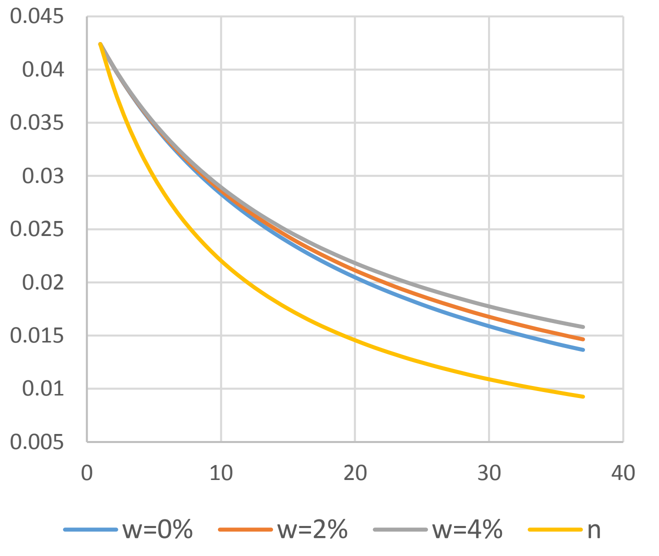
[8] Equation (8) is obtained by applying binomial expansion to (7) and then use the approximations $1+\pi \approx 1+r \approx 1$ and $e-1 \approx e$.

We can obtain the equivalent of (4) from (10) and (12), as below.

$$F = \frac{A - P_{LS}}{P_Y} = \frac{a}{b}\left(\frac{1 + r_\omega}{1 + \pi_\omega}\right)^{N-1} - 3 \tag{13}$$

The funds available in (13) is for each Sterling pound of accrued pension and hence is associated with the same liability given by (6). Based on the preceding analysis, the required real return to achieve full funding will be higher as this includes the more expensive benefits accrued before retirement. Therefore, it can be expected that the required real return $r - \pi$ to achieve full funding will decline with N at a slower rate than is the case for $F_t = L$. This property can be seen from Figure 2 which plots the required real return for life expectancy at retirement e = 23 and static wage rise of ω = 0%. Figure 2 also shows that the higher the wage rise, the more expensive is the pension scheme. This may be understood in terms of r_ω and π_ω in the following way. First, as r_ω may be regarded as an 'average' that takes into account of ω's and r's, one percent rise in r_ω would require more than one percent rise in r so that the rise in F is the same as the rise in F_t. Moreover, for real wage increases, $\pi_\omega > \pi$ which would imply $F < F_t$, others being equal.

Figure 2: Required real return after *N* years of contribution



3. Time diversification of equity risk

For simplicity, the pension portfolio is assumed to comprise of only two types of asset, namely equities and bonds. Equities have higher return whereas bonds have lower risk. This section investigates the trade-off between them. Defining risk as the probability of funds less than liability, it is shown how the risk of equities can be effectively mitigated through the principle of time diversification. The various conditions for how the risk can be reduced are also provided. In particular, when the real interest rate is zero or negative, the risk of underfunding decreases as more funds are allocated to equities.

3.1 Definition of time diversification

The concept of time diversification needs to be clarified as it is surrounded by controversy. To a practitioner, time diversification means that long investment horizons allow investors to hold more risky assets in the portfolio. The definition of time diversification adopted in this paper has a narrower meaning. Consider IID returns $r_1, \dots, r_n$ over n years. Time diversification refers to the property that the average of the n returns has a variance reduced by a factor of n. Such definition is similar to the diversification achieved when more stocks are held in a portfolio. Whether a lower variance would allow more funds to be allocated to risky assets in an optimal portfolio is a separate issue. Below illustrates that utility function is a crucial factor that determines if holding of risky assets should increase for long investment horizon.

Consider the following decreasing relative risk aversion utility function

$$U(W) = \frac{(W - \eta)^{1-\gamma} - 1}{1 - \gamma} \qquad (14)$$

where W here refers to wealth and η and γ are risk-aversion parameters. If $\eta = 0$, the utility function becomes constant relative risk aversion. Thorley (1995) shows that the weight of equity holding in an optimal portfolio is independent of investment horizon. On the other hand, investors with $\eta > 0$ are less risk averse to a percentage loss if they are wealthier. For such investors, the weight of equity holding increases with investment horizon. The case of a defined benefit scheme that promises inflation-linked pension would have $\eta > 0$, for the risk aversion to a percentage loss would be higher if the costs of inflation have not been met. The utility function in (14) is not studied in this paper because no closed form solution can be obtained to show how the risk is dealt with as investment horizon lengthens.[9] Rather, the probability or risk of underfunding for a defined benefit scheme is considered below, as many of such schemes have been closed due to funding crisis.

3.2 Stochastic model

The risk of underfunding is studied first in terms of F_t, the funds available due to a single-year of contribution made n years ago. At the end of this section, it is briefly discussed how the results can be extended to F, the case of a representative pensioner at retirement.

To simplify the notations that shall be used below, subscripts p, e, b and π are used to denote the pension portfolio, equity, bond and inflation respectively. Combination of the subscripts, such as $e - b$, refers to return defined by $r_{e-b} = r_e - r_b$; $e - \pi$ refers to the return $r_{e-\pi} = r_e - \pi$ and so on. Mean and variance are denoted by μ and σ^2 respectively.

Time diversification

Let α be the proportion of funds allocated to the equities in the pension portfolio. To see that the risk of equities in the portfolio can be diversified through long horizon holding,

[9] Thorley (1995) obtains the result by Monte Carlo simulation.

first take logarithm of the compounded returns $\prod_{s=1}^{n}(1+r_s)/(1+\pi_s)$ in (4) to obtain (approximately) sum of n real returns over n years. Next, IID normal assumption of the real returns with mean $\mu_{p-\pi}$ and variance σ_p^2 means that the n-year real return will also have a normal return with mean $n\mu_{p-\pi}$ and variance $n\sigma_p^2$. Therefore, as is shown in Appendix A.1, the probability of underfunding can be obtained as

$$P(F_t < L) = P\left(\sum_{s=1}^{n}\left(\frac{r_s - \pi_s}{1+\pi_s}\right) < l\right) \approx \Phi\left(\frac{l/n - \mu_{p-\pi}}{\sigma_p/\sqrt{n}}\right), \quad (15)$$

where $l = ln(L+3) - ln(a/b)$ can be interpreted as the cost of pension as a result of a single-year contribution made n years ago. L is the liability provided in (6) whereas 3 is the result of the lump sum payment. A higher contribution rate a or a lower accrual rate (larger b) will lower the cost. Dividing by n expresses the cost as a minimum rate the real return on the pension portfolio ($\mu_{p-\pi}$) must exceed.

Perhaps more importantly, the result in (15) reveals that the risk of meeting the pension cost is $\sigma_p/\sqrt{n}$, smaller than the original risk by a factor of $\sqrt{n}$. Let n^* be the value that represents a typical scheme member. Then for an ongoing scheme that is not planned for closure, the risk of meeting the average cost, l/n^*, can be even smaller, others being equal.[10]

Some conditions for risk reduction

Let α be the proportion of funds allocated to equities in a pension portfolio. The analysis below investigates some of the conditions under which changes in α would affect the risk of underfunding. Consider the derivative $\partial z/\partial\alpha$ where z is the argument of the standard normal cumulative distribution function Φ in (15). Since Φ is a monotonic increasing function, a negative (positive) $\partial z/\partial\alpha$ means allocating more funds to equities would lower (raise) the probability of underfunding. Without affecting the conclusion of the analysis, for simplicity, it is assumed that the cost of pension l is a constant. As Appendix A.2 shows, the required derivative can be obtained as below:

$$\frac{\partial z}{\partial\alpha} = -\frac{1}{\sqrt{n}\sigma_p^3}\left\{\sigma_p^2\mu_{e-b} + (\alpha\sigma_{e-b}^2 - \sigma_b^2 + \sigma_e\sigma_b\rho)\mu_{p-\pi}\right\} \quad (16)$$

Now setting $\partial z/\partial\alpha < 0$, for $0 < \alpha < 1$ and $\mu_{b-\pi} \neq 0$, we have

$$\mu_{e-b} > \left\{\frac{\alpha\frac{\sigma_e}{\sigma_b} + (2\alpha+1)\rho}{(1-\alpha)\frac{\sigma_b}{\sigma_e} + \alpha\rho} - 1\right\} \cdot \mu_{b-\pi} \quad (17)$$

If the condition in (17) is satisfied, $\partial z/\partial\alpha$ will be negative, which means that allocating more funds to equities would reduce the risk of underfunding. This will likely be the case if the interest rate is low, making the equity risk premium (μ_{e-b}) large and the real interest rate

[10] The subject of long horizon risk is further discussed in Section 4.

($\mu_{b-\pi}$) low. The empirical evidence provided later in this section suggests that the correlation between equity and bond returns tend to be positive. If we consider the special case where $\rho = 0$, (17) can be further simplified as follows.

$$\mu_{e-b} > \left(\frac{\alpha}{1-\alpha} \frac{\sigma_e^2}{\sigma_b^2} - 1 \right) \cdot \mu_{b-\pi} \tag{18}$$

As α approaches 1, the term in the bracket on the right hand side of (18) will be large. As long as the real interest rate is positive, there exists α beyond which the risk of underfunding will increase, for any given level of equity risk premium.

Illustration

Rates of UK inflation, and returns on world equities and UK long term gilts from 1965 to 2017 are used and the relevant means, standard deviations and correlations are provided in Table 1.

Table 1

	1965-2017	1990-2017	Illustration/simulation
μ_e	0.096	0.080	0.065
μ_b	0.082	0.084	0.020
π	0.051	0.023	0.020
σ_e	0.167	0.153	0.153
σ_b	0.118	0.078	0.078
σ_π	0.049	0.015	0.015
$\rho_{e,b}$	0.288	0.184	0.288
$\rho_{b,\pi}$	-0.022	0.256	-0.022
$\rho_{e,\pi}$	-0.017	-0.021	-0.017

The reported statistics are calculated based on rates of inflation, annual log returns of UK 20-year gilts and world equities denominated in Sterling pound. The world equity returns are calculated using MSCI World Equity Total Return Index.

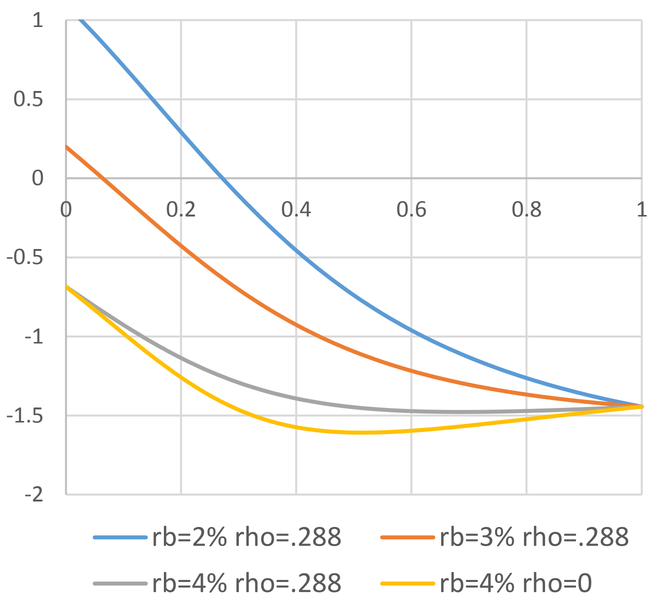
The values in the last column of Table 1 is used for illustration here and then Monte Carlo simulations study later. Note that the illustration assumes a constant rate of inflation and hence the associated standard deviation and correlations (σ_π, $\rho_{b,\pi}$ and $\rho_{e,\pi}$) are assumed to be zero; rates of inflation are treated as stochastic in later simulations study. Also, n = 25 and e = 23 are used for the illustration.

The blue line in Figure 3 below plots the value of z as the equity allocation α increases (note that 'rb' stands for return on bond). Consistent with the preceding analysis, the monotonic decrease of z is mainly due to low real interest rate[11]. With real return on bond increases to 1%, equity premium drops and the rate of falls of z is slower, as is indicated by the orange curve. With interest rate as high as 4%, optimal α locates between 0.6 and 0.8. Finally, when the diversification increases as the correlation between bond and equity declines from 0.288 to 0, z is at its lowest point when the portfolio is a balanced portfolio of

[11] As explained on page 9, falling z means declining risk of underfunding.

half equity and half bond, i.e. α = 0.5. Finally, it is noted that for the case of a representative pensioner with n = 25, the value of z can be as low as -1.5, which implies a below 10% chance of underfunding. As explained above, the risk can be further diversified through long horizon holding for the scheme, as is demonstrated in the even lower probability of underfunding in the simulation study below. Finally, it is noted that time diversification does not play any role in the determination of an optimal α. Also worth noting is that if the contribution rate is increased, optimal α does not change. This further shows that de-risking results in α further away for its optimality; financially speaking, it is unnecessarily expensive.

Figure 3:$(l/n-\mu_{p-\pi})/(\sigma_p/\sqrt{n})$



4 Simulation study

The previous section analyses theoretically the cost and risk of underfunding for the accrued benefit due to a single-year contribution. Section 2.3 suggests that the result can be extended in principle to the case of a pensioner who has made multi-year contribution. The relevant formulas, however, will be long and laborious. This section therefore uses Monte Carlo simulation to study the cost and associated risk of the promised benefit for a representative pensioner as well as for the scheme as a whole. The simulation results are then used to investigate the current disputes surrounding USS at the end of this section.

4.1 A representative pensioner

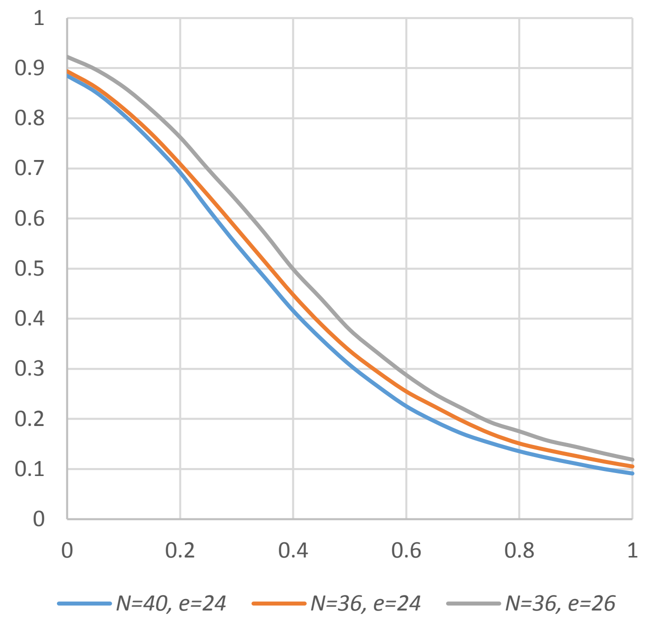
A representative pensioner has contributed for N years and is now at the beginning of retirement which is expected to last for e years. Section 3 uses approximations to obtain the theoretical results. No such approximations are required for the Monte Carlo simulation study and the process of which is described as below.

1. Generate rates of inflation π_s, bond returns $r_{b,s}$ and equity returns $r_{e,s}$ according to the normal distributions specified by column 3 of Table 1
2. Construct portfolio return $r_s = \exp\{\alpha r_{e,s} + (1-\alpha) r_{b,s}\} - 1$
3. Set $\omega_1, \dots, \omega_N = 0.03$
4. Calculate $F = A/P_y - 3$ using equation (9) and (11)

5. Calculate the liability $L = \sum_{s=1}^{e} \prod_{\tau=1}^{s} (1 + \pi_{\tau-1})/(1 + r_\tau)$, where $\pi_0 = 0$

Note that the liability calculated in step 5 takes into account of uncertainty and is more expensive than the liability defined by (6) using static rate of inflation and discount rate. a = 0.222 and b = 1/75 is used for the simulation, which is repeated M = 20,000 times. The frequency of F less than L divided by M is taken as the empirical probability or risk of underfunding. Figure 4 plots the probability of underfunding against α, the proportion of wealth allocated to equities. Consistent with the theoretical analysis in the previous section, it can be seen that as more funds are allocated to equities, the less likely is the chance of underfunding.

Figure 4: P(F<L) for a pensioner



4.2 Extending the analysis to a scheme as a whole

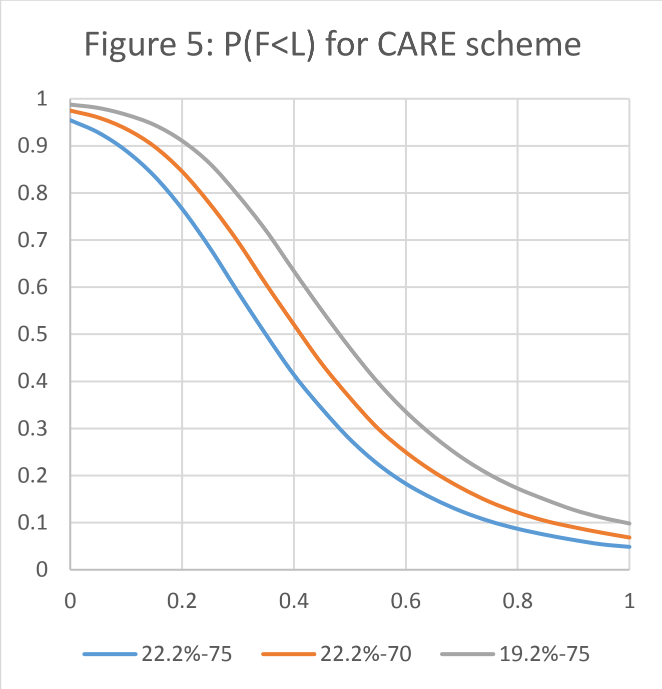
Here, the simulation study is extended to a scheme of representative pensioners. Consider groups G30 to G89; member in Gm is of age m. It is assumed that member retires at the end of age 65. Simulation begins with 1 active member in each of Gm, $30 \leq m \leq 65$. After each year, (i) member in Gm moves to Gm+1; member in G90 'dies'; (ii) a new member age 30 joins G30. 36 years later, a steady state is reached in which the member in G65 has 36 years of accrued pension, the member in G64 has 35 years, and so on until the member in G30 has 1 year. Also, one pensioner in Gm ($66 \leq n \leq 89$); all retiring pensioners have 36 years of accrued pension. For active members who would retire in m years' time, the liability is calculated as

$$L = \left(\frac{1+\pi}{1+r}\right)^m \sum_{s=1}^{e} \prod_{\tau=1}^{s} (1 + \pi_{\tau-1})/(1 + r_\tau), \tag{19}$$

where π and r are the expected rate of inflation and portfolio return respectively. The empirical probability of the sum of all members' funds less than the total liability is calculated based on 20,000 simulations. Figure 5 plots the empirical probability or risk of underfunding against α.

Also, plotted is the other combination of scheme design, such as 19.2%-75 scheme which has a lower contribution rate (from 22.2% to 19.2%) and 22.2%-70 which remains the same contribution rate but raises the accrual rate of promised benefit. It can be seen that in all cases, the defined benefit scheme is affordable, provided there is sufficient equities in the portfolio.

4.3 Discussion of USS



Although limited information is available, the results of the preceding analysis can be used to form educated guesses on the funding position of USS. Three estimates of the required rate of return or discount rate are noted: (i) 3.78% on a prudent basis by Wong (2018); (ii) 3.85% on a breakeven basis by First Actuarial (2017); and (iii) 4.1% on a prudent basis from USS December 2017 report. The simulation exercise in section 4.2 obtains a breakeven discount rate of 4.0% and a prudent discount rate of 4.5%.

These estimates should be lowered for two reasons. First, the simulation assumes there is no cap on pensionable salary whereas USS actually imposes a cap on the pensionable salary at £55,550 (as of March 2018). Second, about one third of USS members are deferred members. These two approximately account for 0.3% bias in the discount rate. After adjustment, the discount rate becomes 3.7% (breakeven) and 4.2% (prudent).

According to its 2017 annual report, about 40% to 50% of USS's assets are invested in equities. The plan is to further de-risk, i.e. replacing equities with bonds. The analysis in section 3 and the simulation study in this section, however, show that de-risking actually increase the risk of underfunding for the scheme. For example, from Figure 5, if de-risking results in a portfolio of 30% equities, then there is 60% chance of underfunding.

It is shown that the risk of holding equities can be diversified through long holding horizon. Is it possible to hold the high yield risky assets without selling during the long time horizon? The answer is a positive 'yes' because USS is still a relatively young scheme. The contributions it receives can pay for the outgoing pension payments. Moreover, there is cash flow generated from the assets such as stock dividends and bond coupons. Indeed, according to First Actuarial (2017), income and liabilities closely match for the next 40 years without touching the assets.

Finally, it must be pointed out that although holding all assets in equities minimises the risk of underfunding, this needs not be the case in practice. A utility function such as the one given in (14) may also be used to determine the appropriate level of allocation to equities.

The current reference portfolio for USS does suggest a sensible equity allocation of between 60% and 70%.

5. Conclusion

The accrued pension and accumulated asset of a representative pensioner are obtained. It is shown that the promised benefit will be cheaper to fund if the pensioner has (i) made more years of contribution; (ii) has become a deferred member; (iii) has a slower wage growth; and (iv) has made the contribution earlier. Consequently, the current CARE scheme is cheaper and less risky than the former final-salary scheme.

The compound growth of assets over a multi-year horizon can be transformed into a sum of real rates of return. The risk of underfunding can then be defined as the likelihood of the average of the real returns to exceed the cost of the promised benefit. Basic statistical theory shows that the longer the time horizon, the more diversification and the lower is the risk of underfunding provided the average real return exceeds the inflation cost. Therefore, de-risking is highly undesirable and is prohibitively expensive given the current low interest rates. Simulation study shows that the estimated cost (expressed as rate of portfolio return or discount rate) is achievable on a prudent basis provided sufficient equities are held in the portfolio.

Reference

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Appendix

A.1

Equating the funds in (4) with the liability L gives

$$\frac{a}{b}\prod_{s=1}^{n}\frac{1+r_s}{1+\pi_s}=L+3.$$

Taking logarithm, we have

$$\sum_{s=1}^{n} ln\left(1+\frac{r_s-\pi_s}{1+\pi_s}\right)=ln(L+3)-ln\left(\frac{a}{b}\right)$$

$$\Rightarrow \sum_{s=1}^{n}\frac{r_s-\pi_s}{1+\pi_s}\approx ln(L+3)-ln\left(\frac{a}{b}\right).$$

Since π_s is relatively small and stable, for simplicity, we use the following approximation

$$\sum_{s=1}^{n}\frac{r_s-\pi_s}{1+\pi_s}\approx\sum_{s=1}^{n}(r_s-\pi)$$

where π is a constant referring to the rate of inflation. Assuming r_s as IID $N\left(\mu_p,\sigma_p^2\right)$, then

$$\sum_{s=1}^{n}\frac{r_s-\pi_s}{1+\pi_s}\approx\sum_{s=1}^{n}(r_s-\pi)\sim N\left(n\mu_{p-\pi},n\sigma_p^2\right)$$

where $\mu_{p-\pi}=\mu_p-\pi$. Let $l=ln(L+3)-ln\left(\frac{a}{b}\right)$. Then

$$P(F_t<L)=P\left(ln\left\{\prod_{s=1}^{n}\frac{1+r_s}{1+\pi_s}\right\}<l\right)\approx\Phi\left(\frac{l-n\mu_{p-\pi}}{\sqrt{n}\sigma_p}\right).$$

A.2

The derivation for equations (16) and (17) in section 3 are provided here. First, $\partial l/\partial\alpha=0$ as l is treated as a constant. Differentiate z with respect to α gives

$$\frac{\partial z}{\partial\alpha}=\frac{\partial}{\partial\alpha}\left(\frac{l/n-\mu_{p-\pi}}{\sigma_p/\sqrt{n}}\right)=-\frac{1}{\sqrt{n}}\left(\frac{1}{\sigma_p}\frac{\partial\mu_p}{\partial\alpha}-\frac{\mu_{p-\pi}}{2\sigma_p^3}\frac{\partial\sigma_p^2}{\partial\alpha}\right).$$

(16) can be obtained by substituting the following derivatives into the above equation.

$$\frac{\partial\mu_p}{\partial\alpha}=\mu_{e-b}$$

$$\frac{\partial\sigma_p^2}{\partial\alpha}=2(\alpha\sigma_{e-b}^2-\sigma_b^2+\sigma_e\sigma_b\rho)$$

To obtain (17), first note that $\mu_{p-\pi} = \alpha\mu_{e-b} + \mu_{b-\pi}$. Set $\partial z/\partial\alpha$ in (16) as negative, which implies

$$\sigma_p^2\mu_{e-b} > (\alpha\mu_{e-b} + \mu_{b-\pi})(\alpha\sigma_{e-b}^2 - \sigma_b^2 + \sigma_e\sigma_b\rho)$$

$$\Rightarrow \mu_{e-b}\left(\sigma_p^2 - \alpha^2\sigma_{e-b}^2 + \alpha\sigma_b^2 - \alpha\sigma_e\sigma_b\rho\right) > \mu_{b-\pi}(\alpha\sigma_{e-b}^2 - \sigma_b^2 + \sigma_e\sigma_b\rho).$$

$$\Rightarrow \mu_{e-b} > \frac{\alpha\sigma_{e-b}^2 - \sigma_b^2 + \sigma_e\sigma_b\rho}{\sigma_p^2 - \alpha^2\sigma_{e-b}^2 + \alpha\sigma_b^2 - \alpha\sigma_e\sigma_b\rho} \cdot \mu_{b-\pi}$$

Substituting $\sigma_p^2 = \alpha^2\sigma_e^2 + (1-\alpha)^2\sigma_b^2 + 2\alpha(1-\alpha)\sigma_e\sigma_b\rho$ and $\sigma_{e-b}^2 = \sigma_e^2 + \sigma_b^2 - 2\sigma_e\sigma_b\rho$ into the above will yield

$$\Rightarrow \mu_{e-b} > \frac{\alpha\sigma_e^2 - (1-\alpha)\sigma_b^2 + (2\alpha+1)\sigma_e\sigma_b\rho}{(1-\alpha)\sigma_b^2 + \alpha\sigma_e\sigma_b\rho} \cdot \mu_{b-\pi}$$

$$\Rightarrow \mu_{e-b} > \left\{\frac{\alpha\sigma_e^2 + (2\alpha+1)\sigma_e\sigma_b\rho}{(1-\alpha)\sigma_b^2 + \alpha\sigma_e\sigma_b\rho} - 1\right\} \cdot \mu_{b-\pi}$$

Divide both the numerator and denominator in the brackets by $\sigma_e\sigma_b$ will give us the required inequality.