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Information Disclosure by a Seller in a Sequential First-Price Auction*

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Abstract

I study a sequential first-price auction where two items are sold to two bidders with private binary valuations. A seller, prior to the second auction, can publicly disclose some information about the outcome of the first auction. I characterize equilibrium strategies for various disclosure rules when the valuations of bidders are either perfectly positively or perfectly negatively correlated across items. I establish outcome equivalence between different disclosure rules. I find that it is optimal for the seller to disclose some information when the valuations are negatively correlated, whereas it is optimal not to disclose any information when the valuations are positively correlated. For most of the parameter values, the seller's revenue is higher if the losing bid is disclosed. When only the winner's identity is disclosed, the equilibrium is efficient whether the valuations are positively or negatively correlated.

Keywords: Efficiency; Information disclosure; Seller's revenue; Sequential first-price auction.

JEL Classification Numbers: D44; D47; D82.

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1 Introduction

Auction houses like Sotheby's and Christie's usually sell multiple items in a sequence on the same auction day. At the Aalsmeer Flower Auction in the Netherlands, many lots of flowers are sold continuously each day. Governments across the world procure goods and services through auctions. Even though a single procurement contract might be awarded in a particular auction, these auctions are often recurring events with the same set of bidders. Because of the sequential nature of auctions in these examples, the auctioneers have a choice how much information to disclose about the outcomes of earlier auctions prior to the start of next auction. I study this question in a setup where an auctioneer sells two items sequentially using a first-price auction to two bidders, whose valuations for each item can take one of two possible values. I consider the following disclosure rules by the seller: 1) disclose both the winning and losing bids of the first auction, 2) only the winning bid, 3) only the losing bid, 4) only the winner's identity, or 5) do not disclose anything. The objective is to identify the disclosure rule that maximizes the seller's revenue and to see how this optimal disclosure rule varies with the prior distribution of valuations.

Although the bidders' valuations of the items are assumed to be independent across bidders, they are correlated across items. Specifically, I consider two extreme cases. The first case considered is when the item valuations are perfectly negatively correlated, meaning, if a bidder has a high value for one item, then he has a low value for the other item and vice versa. The second case considered is when the item valuations are perfectly positively correlated, meaning, a bidder has either high or low values for both items.

By considering these two extreme cases, one can verify how robust are the conclusions about the seller's optimal disclosure rule. The assumption of valuations being perfectly positively correlated is standard in the literature. To justify the case of negative correlation, suppose two paintings, say, one by Rembrandt and the other by Picasso, are auctioned. One can imagine that each participating bidder has strong interest in one of the paintings because it would better complement his collection, but the other bidders are unsure which painting it is.

I establish a set of results. First of all, I find equilibrium bidding strategies in the sequential first-price auction for each combination of disclosure rule and the correlation of valuations (that is, 10 different cases in total). Once the equilibrium strategies are found, I compare the seller's revenue from different information disclosure rules to find the following. If the valuations of items are negatively correlated, then it is beneficial for the seller to disclose some information, whereas if the valuations are positively correlated than any information disclosure harms the seller. The intuition for this result is as follows. If the seller discloses some information from the first auction, it allows the bidders to update their beliefs about the opponent. This, in turn, gives incentives to the bidders to conceal information about their valuations in the first auction. In particular, a bidder who has high value for the second item, benefits if he is perceived by the opponent as someone who has low value for that item because then the bidding in the second auction will be less aggressive. The consequences from concealing their true valuations depend on the correlation of valuations across items. If the valuations are perfectly negatively correlated, then type $(0, 1)$ bidder (that is, low value for the first item and high value for the second item) tries to disguise himself as type $(1,0)$ bidder. Because the latter type bids aggressively for the first item, it means that type $(0,1)$ bidder must also bid relatively aggressively for that item even though his valuation is low for that item. In equilibrium, he even bids above his valuation of the first item with a positive probability. Therefore, the seller benefits from disclosing information from the first auction as it encourages such concealment of valuations by the bidders. On the other hand, when the valuations are perfectly positively correlated, then type $(1,1)$ bidder tries to disguise himself as type $(0,0)$ bidder. Because the latter type does not bid aggressively for the first item, it means that type $(1,1)$ bidder also does not bid too aggressively for that item even though his valuation is high for that item. Consequently, the seller does not benefit from the information disclosure.

I also establish outcome equivalence between various disclosure rules. Interestingly, which rules are outcome equivalent again depends on the correlation between valuations. Thus, if the valuations are negatively correlated, I find that, on one hand, auctions in which either both bids or only the winning bid is disclosed will result in the same equilibrium distribution over outcomes. On the other hand, auctions in which either the winner's identity or the losing bid is disclosed will also lead to the identical equilibrium distribution over outcomes. If, in turn, the valuations are positively correlated, the equivalence classes change. Now I find that, on one hand, auctions in which either both bids or only the losing bid is disclosed are outcome equivalent and, on the other hand, auctions in which either the winner's identity or the winning bid is disclosed are outcome equivalent.

These results can also be related back to the information updating between the first and second auctions and to the properties of the equilibrium in the second auction. At the start of the second auction, each bidder assigns some probability that the opponent has low value for the second item. Lets refer to the bidder who has higher probability of having low value as the weak bidder. Roughly, all that matters for the equilibrium payoffs in the second auction is the probability with which the weak bidder has low value. Next, one can ask which of the two bidders - the winner or the loser of the first auction - is more likely to be the weak bidder. When the valuations are negatively correlated, then between the types $(1, 0)$ and $(0, 1)$, it is likelier that the former type will win the first auction. Therefore, the winner of the first auction will be perceived as the weak bidder in the second auction. This intuitively explains why those disclosure rules that reveal the winner's bid of the first auction are outcome equivalent, on one hand, and why those disclosure rules that do not reveal the winner's bid are outcome equivalent, on the other hand: within each equivalence class, the bidders face identical incentives to conceal their valuations. When the valuations are positively correlated, then between the types $(0, 0)$ and $(1, 1)$, it is likelier that the latter type will win the first auction. Therefore, the loser of the first auction will be perceived as the weak bidder in the second auction. This again explains why those disclosure rules that reveal the loser's bid in the first auction are outcome equivalent, on one hand, and why those disclosure rules that do not reveal the loser's bid are outcome equivalent, on the other hand.

One can also be more specific on the revenue ranking of the disclosure rules. Note that given the equivalence between different disclosure rules, it is enough to compare the revenues from disclosing the winning bid against the revenues from disclosing the losing bid. In the case of negative correlation of valuations, I find that disclosing the winning bid is only optimal when the bidders are of type $(0,1)$ with a relatively high probability. Disclosing the losing bid is always optimal if this probability is less than 71.53%, whereas disclosing the winning bid is always optimal only if this probability is above 93.64%. For the rest of the probability values, the optimal disclosure rule depends on the ratio of marginal valuation (that is, the difference between high and low values) for the second item over the marginal valuation for the first item. The higher is this ratio (which is denoted by v), the more likely it is that the seller will find it optimal to disclose the losing bid. If the valuations are positively correlated, then disclosing the winning bid is only optimal when the bidders are of type $(0,0)$ with a relatively low probability. If this probability exceeds 23.25%, it is always optimal to disclose the losing bid. Though, as already argued before, if the valuations are positively correlated, then any information disclosure is dominated by the information non-disclosure. The ranking of revenues is closely related to the ranking of bidder's payoffs, although the efficiency of the equilibrium outcomes also matters. The results suggest that for most of the parameter values, the bidders bid more aggressively when the losing bid is announced, leading to lower payoffs to the bidders and higher revenue to the seller.

For the case of negatively correlated valuations, I also provide an example of more complicated information disclosure rule that outperforms the disclosure rules considered so far. Specifically, I assume that the seller announces the winning bid if it is below some threshold value, otherwise she announces the losing bid. The equilibrium bidding strategies of the first auction are similar to those when only the losing bid is disclosed, except that the distribution of bids is shifted further to the right. When the seller always discloses the losing bid, there is still uncertainty about the winner's type, even if the loser is perceived as type $(0, 1)$ bidder for sure, and this uncertainty relaxes the competition in the second auction. But now the seller commits to disclosing the winning bid if both bids are relatively low. This reveals that both bidders are likely to be of type $(0, 1)$, resulting in aggressive bidding in the second auction. To avoid it, the bidders bid more aggressively already in the first auction.

There also exists a close relationship between the disclosure rule and the efficiency of the equilibrium outcome in the first auction.[1] I find that the equilibrium is inefficient (resp., efficient) when the bid of the weak bidder is disclosed (resp., not disclosed). Therefore, if the objective of the seller is to implement an efficient outcome irrespective of the sign of correlation between the valuations of the items, she should only announce the winner's identity. If, for example, the seller instead announced the winning bid, then the equilibrium would be inefficient when the valuations are negatively correlated (because the bid of the weak bidder is disclosed), but it would be efficient when the valuations are positively correlated (because the bid of the weak bidder is not disclosed).

To understand this connection between the disclosure rule and efficiency, consider, for example, the case of negatively correlated valuations. Suppose there is an efficient and, consequently, separating equilibrium in the first

[1]The equilibrium outcome of the second auction is always efficient.

auction and both bidders are of type $(0, 1)$. When the winning bid (that is, the bid of the weak bidder) is disclosed, it becomes common knowledge that both bidders are of type $(0, 1)$ and they bid aggressively in the second auction. To avoid it, type $(0, 1)$ wants to pool with the other type and, as a result, the equilibrium is inefficient. On the other hand, when only the losing bid is disclosed, the loser still faces uncertainty about the type of the winner and the bidding in the second auction is not so aggressive. This removes the need for type $(0, 1)$ to pool with the other type. Similar intuition applies to the case with positively correlated valuations.

1.1 Related Literature

I am not the first to study information disclosure in sequential auctions when bidders have multi-unit demand and they know their valuations for all items before any bidding takes place. To the best of my knowledge, however, all the existing studies assume that the valuations across items are perfectly positively correlated. I am the first to explore how the equilibrium strategies and the equilibrium outcomes depend on the sign of correlation between the valuations of the items. This, for example, has allowed to identify the disclosure rule that guarantees the efficiency of the equilibrium whatever is the correlation between the valuations of the items.

The studies closest to this one are Thomas (2010); Cason, Kannan, and Siebert (2011); Kannan (2012) that also investigate the effects of information disclosure in a sequential auction setup with two items and binary valuations.[2] They all, however, assume that the bidders have the same valuations

[2]All these studies consider procurement auctions, but it is inconsequential for the results.

for both items, that is, the valuations are perfectly positively correlated and $v = 1$. Apart from covering the case of negatively correlated valuations, I also contribute to this literature when the valuations are perfectly positively correlated, by allowing v to differ from 1. Thus, for some priors, I find that the revenue is higher if the losing bid is disclosed when v is either close to 0 or 1, but the revenue is higher if the winning bid is disclosed for the intermediate values of v (see Figure 3). Additionally, I consider a wider range of disclosure rules. There are also other studies that use the same setup as the aforementioned studies to explore issues other than information disclosure. Thus, Ding, Jeitschko, and Wolfstetter (2010) study the dynamics of equilibrium prices when both bids are announced; Yao and Xiao (2013) compare the revenues from the simultaneous and sequential auctions when only the winning bid is announced in the latter auctions; Āzacis and Vida (2012) illustrate how the announcement of the winning bid naturally leads to the information exchange between the bidders about their valuations.

Almost all studies that analyse the effects of information disclosure in sequential auctions with multi-unit demand assume binary valuations. Two exceptions are Février (2003) and Tu (2005), which assume that the valuations are drawn from a continuous distribution. Similar to my results, Février (2003) finds that when the valuations are positively correlated, the seller is better off by not disclosing any information than announcing the winner's identity. Tu (2005), similar to this study, considers a range of disclosure rules, but additionally he requires the equilibrium strategies to be monotonically increasing in valuations. This requirement rules out pooling equilibria and even leads to the non-existence of equilibrium for some disclosure rules.

All studies that are cited above, assume a sequence of two auctions. Bergemann and Hörner (2014) depart from this assumption and instead consider an infinite sequence of auctions, but still assume binary and constant valuations. To rule out explicit collusion by bidders, they look for equilibria in Markov strategies and establish that more information hurts both revenue and efficiency. Similar to this study, Bergemann and Hörner (2014) find that the only disclosure rule that ensures that an inefficient equilibrium does not exist is the one when only the winner's identity is disclosed.

Finally, there exists a related literature that compares various disclosure rules in a single-item auction when bidders interact post auction. I mention only few examples. Information disclosure about bids matters in Lebrun (2010) because bidders can engage in resale after the auction; in Giovannoni and Makris (2014) because bidders have reputational concerns; and in Fan, Jun, and Wolfstetter (2016) because bidders engage in oligopolistic competition after a cost-reducing patent has been auctioned to one of them.

The rest of the article is organised as follows. Section 2 sets out the model. Sections 3 and 4 analyse the cases with negative and positive correlations between the valuations of the items, respectively. Section 5 contains concluding discussion. There I show that the equilibrium prices decline in expectation when the valuations are negatively correlated, thus providing another possible explanation for the so-called afternoon effect. Finally, most of the proofs are relegated to the Appendix.

2 The Model

Two bidders are competing for two items in a sequential first-price auction. I assume that each bidder attaches to each item either a low value or a high value. The valuations of one bidder are independent of the valuations of the other bidder. However, bidder's valuations can be correlated across items. The (bivariate Bernoulli) distribution of valuations, which is the same for both bidders, is summarized in the following table:

		Item 2: 0	Item 2: v
Item 1	0	p_{00}	p_{01}
	1	p_{10}	p_{11}

Without loss of generality, the low value is normalized to 0 for both items, and the high value for the first item is normalized to 1. I assume that bidders' payoffs are additive in the two items and money, as well as linear in money.

The items are sold in a sequential first-price auction. The bidders submit sealed bids for the first item and the outcome of this auction is determined. At this point, the seller might disclose some information about the outcome of the first auction. Next, the procedure is repeated for the second item. The bidder who submits the highest bid for an item, obtains it and pays his bid for that item. If there is a tie in either auction, I assume that it is broken randomly, except in the second auction when bidders have different valuations. In that case, I assume that the second item is assigned to the bidder with higher valuation.[3] Finally, I do not require the bids to be non-

[3]Alternatively, given that I will only consider the cases when the valuations are either perfectly positively or perfectly negatively correlated, one can assume that the priority in the second auction is given to the winner of the first auction in the former case, that is, when the valuations are positively correlated, whereas the priority is given to the loser of

negative.

Before the start of the second auction, an auctioneer has a choice how much information to release about the outcome of the first auction. The auctioneer can choose to disclose both the winning and losing bids, only the winning bid, only the losing bid, only the winner's identity, or she can choose not to disclose anything.[4] The objective is to derive equilibrium strategies and to compare the seller's revenue from the sequential auction under different disclosure rules. I adopt the perfect Bayesian equilibrium as the solution concept, but additionally I also require that no bidder uses dominated strategies in the equilibrium.[5] The out-of-equilibrium beliefs will be specified as follows: if the equilibrium bid of a bidder in the first auction must belong to an interval $\left[\underline{b}, \overline{b}\right]$, but instead it is below $\underline{b}$ (resp., above $\overline{b}$), then the beliefs about this bidder will be exactly the same as the ones if he had bid $\underline{b}$ (resp., $\overline{b}$).

Rather than solving for the equilibrium strategies for all possible distributions of valuations, in the continuation, I consider two extreme cases: when the valuations are perfectly negatively correlated, and when they are perfectly positively correlated. These cases correspond to $p_{00} = p_{11} = 0$ and $p_{01} = p_{10} = 0$, respectively.

the first auction in the latter case. Intuitively, in both cases, the priority is given to the bidder who is more likely to have the high value for the second item.

[4]More precisely, when either or both bids are disclosed, I additionally assume that each bidder also learns whether or not he has won the first auction. This additional assumption is only needed when ties arise with a strictly positive probability in the equilibrium of the first auction.

[5]As a result of information disclosure, it can become common knowledge that one of the bidders has high value for the second item. Then, there exists an equilibrium in the second auction, in which both bidders submit bids equal to the high value even if one of them has low value. I rule out such equilibria.

3 Perfectly Negatively Correlated Valuations

I first consider the case when $p_{00} = p_{11} = 0$. To simplify notation, let us denote p_{01} by p, and p_{10} by $1-p$. I also assume that $p \in (0,1)$.

I start by stating the equilibrium strategies in the second auction. Let w and l denote the winner and the loser of the first auction, respectively. For $i = w, l$, let q_i be the probability that bidder i is of type $(1,0)$. As mentioned in the Introduction, when the valuations are negatively correlated, the winner of the first auction is perceived to be type $(1,0)$ bidder with higher probability. Therefore, I assume that $q_w \geq q_l$. Later I will verify that this assumption is indeed satisfied.

The following lemma describes the equilibrium strategies of the second auction whenever the probabilities q_i for $i = w, l$ are common knowledge between the opponents and $q_w \geq q_l$.

Lemma 1 *Type* $(1,0)$ *bids* 0. *Type* $(0,1)$ *bids* v *if* $q_w = 0$, *and bids* 0 *if* $q_w = 1$. *If* $0 < q_w < 1$, *then*

1. *Bidder* w *of type* $(0,1)$ *draws a bid* c *according to the distribution function*
$$G_w(c) = \frac{q_w}{1-q_w}\frac{c}{v-c} \tag{1}$$
on the interval $[0,(1-q_w)v]$,

2. *Bidder* l *of type* $(0,1)$ *draws a bid* c *according to the distribution function*
$$G_l(c) = \frac{q_w}{1-q_l}\frac{v}{v-c} - \frac{q_l}{1-q_l} \tag{2}$$

on the interval $[0,(1-q_w)v]$*, and puts a mass* $G_l(0)>0$ *on bid* 0 *if* $q_w>q_l$.

The equilibrium payoffs of types $(1,0)$ *and* $(0,1)$ *are, respectively, 0 and* $q_w v$ *in the second auction.*

Because versions of this lemma have already been derived in the literature (see, for example, Maskin and Riley (1985)), the proof of Lemma 1 is omitted.

Let $F_{01}(b)$ and $F_{10}(b)$ denote the equilibrium strategies of types $(0,1)$ and $(1,0)$, respectively, in the auction for the first item.

3.1 When No Information Is Disclosed

I start by characterizing the equilibrium strategies when no information is disclosed. This case will serve as a benchmark.

Proposition 2 Item 1: *Type* $(0,1)$ *bids 0 and type* $(1,0)$ *draws a bid according to* $F_{10}:[0,1-p]\rightarrow[0,1]$*, where*

$$F_{10}(b)=\frac{p}{1-p}\frac{b}{1-b}.$$

Item 2: *The bidders bid as specified in Lemma 1, where* $q_w=q_l=1-p$*.*[6]

The equilibrium payoffs of types $(1,0)$ *and* $(0,1)$ *are, respectively,* $\pi_{10}=p$ *and* $\pi_{01}=(1-p)v$*.*

Because no information is disclosed between the two auctions, they can be treated as two independent auctions. As a result, the equilibrium bidding

[6] Because $q_w=q_l$, it does not matter that the bidder does not know whether he is the winner or the loser of the first auction.

in the first auction is similar to the one described in Lemma 1 (with the strategies of types $(1,0)$ and $(0,1)$ exchanged, and setting $v = 1$ and $q_w = q_l = p$).

The equilibrium of the sequential auction is efficient with the surplus equal to $(1-p)^2 + p^2 v + 2p(1-p)(1+v)$, the ex ante payoff of a bidder is $p(1-p)(1+v)$, and the equilibrium revenue of the seller is equal to $(1-p)^2 + p^2 v$.

3.2 Only The Winning Bid Is Disclosed

Before stating the equilibrium strategies of the first auction formally, I describe them in words. The equilibrium strategies are also illustrated for specific parameter values in Figure 1. Type $(0,1)$ bidder bids 0 with a positive probability, which is less than 1. With the remaining probability, he bids above 0, i.e., above his valuation of the first item. Type $(1,0)$ bids above 0 with probability 1. Depending on the parameter values, either both types randomize on the same interval or the support of type $(0,1)$ bids is a strict subset of the support of type $(1,0)$ bids. Because type $(0,1)$ can win the first auction against type $(1,0)$ with a strictly positive probability, the equilibrium is inefficient.

Proposition 3 Item 1: $F_{01} : \left[0, \bar{b}\right] \to [0,1]$, *where*

$$F_{01}(b) = F_{01}(0) \frac{v + b + \ln(1-b)}{v(1-b)}.$$

If $pv \geq 1 - e^{-v}$, *then* $F_{10} : \left[0, \bar{b}\right] \to [0,1]$, *where*

$$F_{10}(b) = -F_{01}(0) \frac{p}{1-p} \frac{b + \ln(1-b)}{v(1-b)}, \tag{3}$$

$F_{01}(0) = \frac{1-\bar{b}}{p}$ and $\bar{b}$ is given by $v(1-p) + \bar{b} + \ln\left(1-\bar{b}\right) = 0$.

If $pv < 1 - e^{-v}$, then $F_{10} : \left[0, 1 - \frac{pv}{e^v - 1}\right] \to [0,1]$, where $F_{10}(b)$ is defined in (3) for $b \in \left[0, \bar{b}\right]$, and it is

$$F_{10}(b) = F_{01}(0) \frac{p}{1-p} \frac{1}{1-b} - \frac{p}{1-p} \tag{4}$$

for $b \in \left(\bar{b}, 1 - \frac{pv}{e^v - 1}\right]$, $F_{01}(0) = \frac{v}{e^v - 1} < 1$ and $\bar{b} = 1 - e^{-v}$.

Item 2: *The bidders bid as specified in Lemma 1, where for $0 \leq b_w \leq \bar{b}$,*[7]

$$\begin{aligned} q_w &= \frac{(1-p) f_{10}(b_w)}{p f_{01}(b_w) + (1-p) f_{10}(b_w)} = -\frac{\ln(1-b_w)}{v}, \\ q_l &= \frac{(1-p) F_{10}(b_w)}{p F_{01}(b_w) + (1-p) F_{10}(b_w)} = -\frac{b_w + \ln(1-b_w)}{v}. \end{aligned}$$

The equilibrium payoffs of types $(1,0)$ and $(0,1)$ are, respectively, $\pi_{10} = pF_{01}(0)$ and $\pi_{01} = (1-p)v$.

The proof of this and other propositions that are omitted from the main text, can be found in the Appendix.

The ex ante payoff of a bidder is

$$p\pi_{01} + (1-p)\pi_{10} = p(1-p)(v + F_{01}(0)).$$

Thus, the payoff is lower compared with the situation when no information is released. The expected surplus from the auction is

$$\begin{aligned} &(1-p)^2 \times 1 + p^2 \times 0 + 2p(1-p) \times \int_0^{\bar{b}_{10}} F_{01}(b) f_{10}(b)\, db \\ &+ (1-p)^2 \times 0 + p^2 \times v + 2p(1-p) \times v, \end{aligned}$$

[7] When $b_w = 0$, $q_w = \frac{(1-p) f_{10}(0)}{p F_{01}(0) + (1-p) f_{10}(0)} = 0$. If $b_w > \bar{b}$, then set $q_w = \frac{(1-p) f_{10}\left(\bar{b}\right)}{p f_{01}\left(\bar{b}\right) + (1-p) f_{10}\left(\bar{b}\right)}$.

where the first line is the surplus from the first auction and the second line is the surplus from the second auction. Clearly, because the equilibrium is inefficient, the surplus of the first auction is not maximized. The expected revenue of the seller is the difference between the surplus and the bidders' payoffs, and is given by

$$R_n^w = (1-p)^2 + p^2 v + 2p(1-p)\left(\int_0^{\bar{b}_{10}} F_{01}(b)\, f_{10}(b)\, db - F_{01}(0)\right).$$

Because $F_{01}(b) \geq F_{01}(0)$ for all b, the expected revenue exceeds the one under no disclosure.

3.3 The Winning and Losing Bids Are Disclosed

It turns out that in this case, the equilibrium payoffs of the bidders and the equilibrium revenue of the seller are the same as in the case when only the winning bid is disclosed. Even more, the equilibrium strategies of both types in the first auction are exactly the same for both disclosure rules. The only change in the strategy is for type $(0,1)$ bidder in the second auction after he has lost the first auction. The intuition for this result is simple. The expected payoff of type $(0,1)$ bidder in the second auction only depends on the value of q_w. Whether or not the losing bid of the first auction is also disclosed does not affect this value. Therefore, the first auction's strategies are unaffected by the decision to disclose the losing bid in addition to the winning bid.

Proposition 4 *The equilibrium strategies are the same as in Proposition 3,* *except that for* $0 \leq b_l \leq \bar{b}$,[8]

$$q_l = \frac{(1-p) f_{10}(b_l)}{p f_{01}(b_l) + (1-p) f_{10}(b_l)} = -\frac{\ln(1-b_l)}{v}.$$

Proof. One only needs to verify two things. First, the winner's belief about the opponent's type at the start of the second auction, as described by q_l, is indeed derived from the strategies of the first auction using Bayes' formula. Second, the requirement that $q_l \leq q_w$ is also satisfied. ■

3.4 Only The Losing Bid Is Disclosed

The equilibrium bidding strategies in this and next section are only valid for $v \leq 1$. Given this restriction, one can verify that $-\ln(1-p)\frac{1-p}{p}v < (1-p)(1-\ln(1-p)v)$ or, equivalently, $p + (1-p)\ln(1-p)v > 0$ holds. Likewise, $(1-p)v < -\ln(1-p)\frac{1-p}{p}v$ or, equivalently, $p + \ln(1-p) < 0$ holds. Therefore, the supports of the equilibrium strategies in the following proposition are well-defined.

Proposition 5 *Suppose* $v \leq 1$.

Item 1: $F_{01} : \left[(1-p)v, -\ln(1-p)\frac{1-p}{p}v\right] \to [0,1]$, *where* F_{01} *is implicitly defined by*

$$pF_{01}(b)(-b) = (1-p)\ln(1-pF_{01}(b))\,v. \tag{5}$$

$F_{10} : \left[-\ln(1-p)\frac{1-p}{p}v, (1-p)(1-\ln(1-p)v)\right] \to [0,1]$, *where*

$$F_{10}(b) = \frac{p}{1-p}\frac{b + \ln(1-p)\frac{1-p}{p}v}{1-b}.$$

[8] If $b_l > \bar{b}$, then $q_l = \frac{(1-p)f_{10}(\bar{b})}{pf_{01}(\bar{b})+(1-p)f_{10}(\bar{b})}$.

Item 2: *The bidders bid as specified in Lemma 1, where* $q_l = 0$ *if* $b_l \leq -\ln(1-p)\frac{1-p}{p}v$ *and* $q_l = 1$ *if* $b_l > -\ln(1-p)\frac{1-p}{p}v$*, whereas*

$$q_w = \frac{1-p}{1-p+p\left(1-F_{01}\left(b_l\right)\right)}. \tag{6}$$

The equilibrium payoffs of types $(1,0)$ *and* $(0,1)$ *are, respectively,* $\pi_{10} = p + (1-p)\ln(1-p)v$ *and* $\pi_{01} = (1-p)v$*.*

In words, in the first auction, types $(0,1)$ and $(1,0)$ randomize on adjacent intervals. Therefore, the type of the loser of the first auction (and sometimes the type of the winner) is fully revealed prior to the second auction.[9] Also note that the lowest possible equilibrium bid in the first auction is $(1-p)v$. Thus, type $(0,1)$ bids strictly above his valuation for the first item.

Given the equilibrium strategies, the ex ante payoff of a bidder is

$$p\pi_{01} + (1-p)\pi_{10} = p(1-p)(1+v) + (1-p)^2\ln(1-p)v.$$

Because the equilibrium of the auction is efficient, the surplus from the auction is $(1-p)^2 + p^2v + 2p(1-p)(1+v)$. The expected revenue of the seller is

$$R_n^l = (1-p)^2 + p^2v - 2(1-p)^2\ln(1-p)v.$$

3.5 Only The Winner's Identity Is Disclosed

Now, when deriving the equilibrium strategies, one cannot apply the results of Lemma 1 to the second auction because the values of q_w and q_l will not be

[9] If type $(0,1)$ is the loser of the first auction, then there exists an equilibrium in the second auction, where both bidders submit a bid equal to v irrespective of the type of the winner of the auction. As mentioned before, I rule out such equilibria as they involve using weakly dominated strategies. I do it by requiring type $(1,0)$ to bid 0 in Lemma 1.

common knowledge between the bidders. For example, the belief of the loser of the first item that the opponent has low valuation for the second item will depend on the loser's bid, which is known to him but unknown to the opponent. In order to define the equilibrium strategies of the second auction in the proposition below, I introduce the following function:

$$c(b) = \frac{p(1 - F_{01}(b))}{1 - p + p(1 - F_{01}(b))} v, \tag{7}$$

where $F_{01}(b)$ is defined in (5) and b takes values in the interval $\left[(1-p)v, -\ln(1-p)\frac{1-p}{p}v\right]$. This function will define a mapping from the bids b of the first auction into the bids c of the second auction. To understand its properties, let $H_w(c) \equiv \frac{1-p}{p}\frac{c}{v-c}$ for $c \in [0, pv]$. Then, one can verify that $1 - F_{01}(b) = H_w(c(b))$ holds. Further, because $H_w'(c) > 0$ and $F_{01}'(b) > 0$, it follows that $c'(b) < 0$. Thus, $c(b)$ is monotonically decreasing in b, and takes values in the interval $[0, pv]$.[10]

Proposition 6 *Suppose $v \leq 1$.*

Item 1: *The bidders bid as in Proposition 5.*

Item 2: *Type $(1,0)$ bids 0. Bidder w of type $(0,1)$ bids c_w which is defined by (7) for $b = b_w$. Bidder l of type $(0,1)$ draws a bid c from the interval $[0, (1-q_w)v]$ according to*

$$H_l(c) = \frac{q_w v}{v - c}$$

[10] Also, the inverse of function $c(b)$ is

$$b(c) = -\ln\left(\frac{(1-p)v}{1-c}\right)\frac{(1-p)(v-c)}{pv-c}v.$$

where q_w is defined in (6).[11] *The equilibrium payoffs of types* $(1, 0)$ *and* $(0, 1)$ *are, respectively, 0 and $q_w v$ in the second auction.*

Thus, in the equilibrium of the second auction, type $(0, 1)$ loser of the first auction draws a random bid from an interval (as is usual in the first-price auction with binary valuations), but type $(0, 1)$ winner of the first auction submits a bid, which is a monotone transformation of his bid in the first auction. Though, as shown in the proof, from the perspective of type $(0, 1)$ loser, it is as if type $(0, 1)$ winner is randomizing on the interval $[0, (1 - q_w)v]$ according to $\frac{H_w(c)}{H_w((1-q_w)v)}$ in the second auction. As a result, type $(0, 1)$ bidder, either he wins or loses the first auction, expects the same payoff of $q_w v$ in the second auction (which he could guarantee by submitting a bid of $(1 - q_w)v$). Interestingly, the winner of the first auction does not know the value of q_w. However, his strategy in the second auction does not depend on this value. Furthermore, because the expression for q_w is the same as the one when only the losing bid is disclosed, the equilibrium strategies of the first auction are also the same under both disclosure rules. As a result, the equilibrium outcome of the sequential auction when only the winner's identity is disclosed, is also equivalent to the one when only the losing bid is disclosed.[12]

[11] If type $(0, 1)$ submitted a bid $b \notin \left[(1 - p)v, -\ln(1 - p)\frac{1-p}{p}v\right]$ in the first auction, then he bids in the second auction as if he had submitted $b = (1 - p)v$ or $b = -\ln(1 - p)\frac{1-p}{p}v$, whichever is closer to his actual bid in the first auction.

[12] One can also note that $\frac{H_w(c)}{H_w((1-q_w)v)} = G_w(c)$, which is given in (1), and $H_l(c) = G_l(c)$, which is given in (2) when $q_l = 0$. Thus, the equilibrium distribution of bids in the second auction is the same as the one implied by Lemma 1.

3.6 The Ranking of Revenues

Given the outcome equivalence between various disclosure rules, it is enough to compare the cases when only the winner's bid or only the loser's bid is disclosed (when $v \leq 1$). The revenue from the auction when the winner's bid is disclosed is higher than the revenue when only the loser's bid is disclosed, $R_n^w \geq R_n^l$ if

$$\Delta_n = p\left(\int_0^{\bar{b}_{10}} F_{01}(b) f_{10}(b)\, db - F_{01}(0)\right) + (1-p)\ln(1-p)\, v \geq 0.$$

When $pv \geq 1 - e^{-v}$, the integral in the above expression is given by

$$-\frac{(1-\bar{b})^2}{p(1-p)v^2}\int_0^{\bar{b}} \frac{(v+b+\ln(1-b))\ln(1-b)}{(1-b)^3} db = \frac{\bar{b}^2 + 2p(1-p)v}{4p(1-p)v},$$

where I have used the relationship $\ln\left(1-\bar{b}\right) = -\bar{b} - (1-p)v$. Hence,

$$\Delta_n = \frac{\bar{b}^2 + 2p(1-p)v}{4(1-p)v} - \left(1-\bar{b}\right) + (1-p)\ln(1-p)\, v,$$

and $\bar{b}$ is given by $v(1-p) + \bar{b} + \ln\left(1-\bar{b}\right) = 0$.

When $pv < 1 - e^{-v}$, the integral is given by

$$\begin{aligned} & 1 - F_{10}\left(1-e^{-v}\right) + \int_0^{1-e^{-v}} F_{01}(b) f_{10}(b)\, db \\ = \ & 1 + \frac{p}{1-p}\frac{1-v-e^{-v}}{1-e^{-v}} - \frac{p}{1-p}\frac{1}{(e^v-1)^2}\int_0^{1-e^{-v}} \frac{(v+b+\ln(1-b))\ln(1-b)}{(1-b)^3} db \\ = \ & 1 + \frac{p}{1-p}\frac{e^v - ve^v - 1}{e^v - 1} - \frac{1}{4}\frac{p}{1-p}\frac{-v - 4e^v + 2e^{2v} - 3ve^{2v} + 4ve^v + 2}{(e^v-1)^2}. \end{aligned}$$

Hence,

$$\Delta_n = p\left(1 + \frac{3pv - 4v - 2p + 2pe^v - pve^v}{4(e^v-1)(1-p)}\right) + (1-p)\ln(1-p)\, v.$$

$\Delta_n = 0$ is plotted in Figure 2. For the (p, v) pairs to the right of $\Delta_n = 0$ curve, it is optimal for the seller to disclose the winning bid, and for the (p, v) pairs to the left of $\Delta_n = 0$ curve, it is optimal to disclose the losing bid. Further, if $p \leq p^* \approx 0.7153$, then it is not optimal to disclose the winning bid for any value of $v \leq 1$. The reason why R_n^l is higher for most of the parameter values, can be partly attributed to the inefficiency of the equilibrium when the winning bid is disclosed. The main reason, however, is that for most parameter values, the bidders appear to bid more aggressively and, consequently, expect a lower payoff when the losing bid is disclosed. This is shown in Figure 2 by $\delta_n = 0$, where $\delta_n = -\left(p\,(1-p)\,(1 - F_{01}\,(0)) + (1-p)^2 \ln(1-p)\,v\right)$ measures the difference in ex ante payoffs of a bidder when the winning bid and the losing bid are, respectively, disclosed. For the (p, v) pairs to the left of $\delta_n = 0$ curve, bidder's payoff is higher when the seller discloses the winning bid.

3.7 Other Disclosure Rules

So far I have studied simple, "natural" disclosure rules but, in principle, the seller could also adopt a more sophisticated policy. Namely, she could disclose whether or not the submitted bids (b_w, b_l) belong to some set. I now provide an example that shows that the seller can increase her revenue even further by adopting such more sophisticated disclosure rule. The disclosure rule that I consider is as follows. The seller announces that the winning bid of the first auction is b_w if $b_w < k$, but she announces that the losing bid is b_l if $b_w \geq k$. To emphasize, the seller does not just announce a number. She also reveals whether she announces the winning or losing bid. Said differently, she reveals whether or not the bids (b_w, b_l) belong to the set $B = \{(b_w, b_l) | b_w < k\}$.

To simplify the exposition, here I set $v = 1$.

Proposition 7 *Suppose k is such that $1 - p \in \left[\frac{k^2}{1+k\ln(k)}, k\right]$ holds.*

Item 1: $F_{01} : \left[0, \bar{b}_{01}\right] \to [0,1]$, *where* $F_{01}(b) = \frac{k+p-1}{kp}$ *for* $b \in [0,k]$, *and* $F_{01}(b)$ *for* $b \in \left(k, \bar{b}_{01}\right]$ *is implicitly defined by*

$$pF_{01}(b)(-b) + pF_{01}(k)k = (1-p)\left(\ln\left(1 - pF_{01}(b)\right) - \ln\left(1 - pF_{01}(k)\right)\right).$$

$F_{10} : \left[\bar{b}_{01}, \bar{b}_{10}\right] \to [0,1]$, *where*

$$F_{10}(b) = \frac{p}{1-p} \frac{b - \bar{b}_{01}}{1-b},$$

and $\bar{b}_{01} = \frac{k+p-1-(1-p)\ln(k)}{p}$ *and* $\bar{b}_{10} = 1 - p + p\bar{b}_{01}$.

Item 2: *The bidders bid as specified in Lemma 1, where* $q_l = q_w = 0$ *if* $(b_w, b_l) \in B$, *otherwise* $q_l = 0$ *if* $b_l \le \bar{b}_{01}$ *and* $q_l = 1$ *if* $b_l > \bar{b}_{01}$, *and* q_w *is defined as in (6).*

The equilibrium payoffs of types $(1,0)$ *and* $(0,1)$ *are, respectively,* $\pi_{10} = p(1 - \bar{b}_{01})$ *and* $\pi_{01} = 1 - p$.

In words, type $(0,1)$ bidder bids either 0 with a positive probability,[13] or randomizes on the interval $[k, \bar{b}_{01}]$, and type $(1,0)$ randomizes on the adjacent interval $[\bar{b}_{01}, \bar{b}_{10}]$. Thus, interestingly, none of the types bids in $(0,k)$. The bidding strategies are similar to the ones when only the losing bid is disclosed. In fact, if one assumed that $k < 1-p$, then the equilibrium would be exactly the same as the one derived in Proposition 5. In that proposition, type $(0,1)$

[13] Therefore, to ensure that $F_{01}(0) \ge 0$, I require that $k \ge 1 - p$.

randomizes on the interval $[1-p, \bar{b}_{01}]$. Roughly speaking, under the current disclosure rule, the original probability that the bid of type $(0,1)$ lies in the interval $[1-p, k]$ is being shifted to bid 0. However, additionally, higher values of k push $\bar{b}_{01}$ to the right. Although type $(0,1)$ still expects the same payoff as before, type $(1,0)$ must now bid more aggressively because of higher $\bar{b}_{01}$ and, as a result, expects smaller payoff. This, in turn, means higher revenue for the seller.

Given the equilibrium strategies, the ex ante payoff of a bidder is

$$p\pi_{01} + (1-p)\,\pi_{10} = p\,(1-p)\,(2-\bar{b}_{01}).$$

Because the expected surplus is $(1-p)^2 + p^2 + 4p\,(1-p)$, the expected revenue of the seller is

$$R_n^k = (1-p)^2 + p^2 + 2p\,(1-p)\,\bar{b}_{01},$$

which is clearly increasing in $\bar{b}_{01}$. On the other hand, $\bar{b}_{01}$ is increasing in k (when $k \geq 1-p$). Thus, higher values of k result in higher revenues for the seller. Note that for $k = 1-p$, the seller's revenue is exactly R_n^l. Hence, by setting k such that $k > 1-p \geq \frac{k^2}{1+k\ln(k)}$ holds, it is true that $R_n^k > R_n^l$. For $p < 0.9364$ (see Figure 2), this disclosure rule also dominates the one in which only the winning bid is announced, $R_n^k > R_n^w$.

4 Perfectly Positively Correlated Valuations

I now consider the case when $p_{01} = p_{10} = 0$. As noted in the Introduction, this case has already been studied in the literature for some disclosure rules although assuming $v = 1$. To simplify notation, let us now denote p_{00} by

p, and p_{11} by $1-p$. I again assume that $p \in (0,1)$. Below I am going to establish a similar set of results as in the case of negatively correlated values.

As before, let w and l be the winner and the loser of the first auction, respectively. For $i = w, l$, let q_i now be the probability that bidder i is of type $(0,0)$. Next, I restate Lemma 1 but now I assume that $q_w \leq q_l$. That is, I assume that the loser of the first auction is perceived to be type $(0,0)$ bidder with higher probability. I will verify it once the strategies for the first auction are stated.

Lemma 8 *Type* $(0,0)$ *bids* 0*. Type* $(1,1)$ *bids* v *if* $q_l = 0$*, and bids* 0 *if* $q_l = 1$*. If* $0 < q_l < 1$*, then*

1. *Bidder* w *of type* $(1,1)$ *draws a bid* c *according to the distribution function*
$$G_w(c) = \frac{q_l}{1-q_w}\frac{v}{v-c} - \frac{q_w}{1-q_w} \tag{8}$$
on the interval $[0,(1-q_l)v]$*, and puts a mass* $G_w(0) > 0$ *on bid* 0 *if* $q_w < q_l$*.*

2. *Bidder* l *of type* $(1,1)$ *draws a bid* c *according to the distribution function*
$$G_l(c) = \frac{q_l}{1-q_l}\frac{c}{v-c} \tag{9}$$
on the interval $[0,(1-q_l)v]$*.*

The equilibrium payoffs of types $(0,0)$ *and* $(1,1)$ *are, respectively, 0 and* $q_l v$ *in the second auction.*

For all the disclosure rules that I am going to consider, it is true that type $(0,0)$ bids 0 in the equilibrium of the first auction.[14] That it is indeed an

[14] Note that I allow the bids to be negative.

equilibrium strategy will follow from the fact that the lowest bid that type $(1,1)$ submits in equilibrium, is more or equal to 0. Therefore, I will omit the strategy of type $(0,0)$ bidder from the propositions that follow. I also denote the equilibrium strategy of type $(1,1)$ in the first auction by $F(b)$.

4.1 When No Information Is Disclosed

The equilibrium strategies are exactly the same as in Proposition 2 with types $(0,1)$ and $(1,0)$ replaced by types $(0,0)$ and $(1,1)$, respectively, and $q_l = q_w = p$. The equilibrium is efficient, the expected surplus is $(1-p^2)(1+v)$, the ex ante payoff of a bidder is $p(1-p)(1+v)$, and the equilibrium revenue of the seller is equal to $(1-p)^2(1+v)$.

4.2 Only The Winning Bid Is Disclosed

Here I assume that $v \leq 1$.

Proposition 9 *Suppose $v \leq 1$.*

Item 1: $F : [0, 1-p+vp\ln p] \to [0,1]$, *where F is implicitly defined by*[15]

$$(p + (1-p)\,F(b))\,(1-b) - vp\ln(p + (1-p)\,F(b)) = p\,(1 - v\ln p)\,. \tag{10}$$

Item 2: *The bidders bid as specified in Lemma 8, where $q_l = q_w = 1$ if $b_w = 0$, otherwise $q_w = 0$, and*[16]

$$q_l = \frac{p}{p + (1-p)\,F(b_w)}. \tag{11}$$

[15]That F is monotonically increasing is verified in the next subsection. There I need the assumption that $v \leq 1$.

[16]If $b_w > 1-p+vp\ln p$, then $q_l = p$.

The equilibrium payoff of type $(1,1)$ *is* $p(1+v\,(1-\ln p))$.

One can verify that $F\,(0)=0$, which means that the equilibrium is efficient and the surplus is $(1-p^2)(1+v)$. The ex ante equilibrium payoff of a bidder is $p\,(1-p)\,(1+v\,(1-\ln p))$. Hence, the seller's revenue is given by

$$R_p^w=(1-p)^2(1+v)+2p(1-p)v\ln p.$$

It follows that the bidders gain and the seller loses compared to the case when no information is revealed.

4.3 Only The Winner's Identity Is Disclosed

Because the values of q_i for $i=w,l$ are not common knowledge between the bidders when only the winner's identity is disclosed, one cannot apply the results of Lemma 8 to the second auction. In order to define the equilibrium strategies of the second auction, I introduce the following function:

$$b\,(c)=\frac{c}{v}+(v-c)\ln\left(1-\frac{c}{v}\right) \tag{12}$$

for $c\in[0,(1-p)v]$. It will define an implicit mapping from the bids b of the first auction into the bids c of the second auction. First, observe that the function in (12) is monotonically increasing,

$$\frac{db}{dc}=\frac{1-v}{v}-\ln\left(1-\frac{c}{v}\right)>0$$

given that $0\leq c<v\leq 1$, and b takes values in $[0,1-p+vp\ln p]$. Second, if one defines a distribution function $H_l\,(c)\equiv\frac{p}{1-p}\frac{c}{v-c}$ for $c\in[0,(1-p)v]$, then one can verify that $F(b(c))=H_l(c)$ holds for F defined in (10). (Also,

$F'(b)\, b'(c) = H_l'(c)$. Because $b'(c) > 0$ and $H_l'(c) > 0$, then indeed $F'(b) > 0$ in (10) as required.) From $F(b) = \frac{p}{1-p}\frac{c}{v-c}$, one can rewrite (12) as

$$c = \frac{(1-p)\, F(b)}{p + (1-p)\, F(b)} v. \tag{13}$$

Proposition 10 *Suppose $v \leq 1$.*

Item 1: *The bidders bid as in Proposition 9.*

Item 2: *Type $(0,0)$ bids 0. Bidder l of type $(1,1)$ bids c_l, which is defined by (13) for $b = b_l$. Bidder w of type $(1,1)$ draws a bid c from the interval $[0, c_w]$ according to*

$$H_w(c) = \frac{v - c_w}{v - c}$$

where c_w is defined by (13) for $b = b_w$.[17] *The equilibrium payoffs of types $(0,0)$ and $(1,1)$ are, respectively, 0 and $v - c_w = q_l v$ in the second auction where q_l is given by (11).*

Note that the winner of the first auction does not know what c_l is, but he knows that it is less than c_w and that the distribution of c_l is $\frac{H_l(c_l)}{H_l(c_w)}$, provided that the opponent's type is $(1,1)$. Therefore, from the perspective of the winner, type $(1,1)$ loser effectively randomizes as specified in Lemma 8. Because type $(1,1)$ winner also randomizes as specified in Lemma 8 (where $q_w = 0$), the strategies in the second auction form an equilibrium, in which both expect a payoff of $v - c_w = q_l v$. Further, conditional on the bids of the first auction, the value of q_l is the same whether the winning bid or the

[17] If type $(1,1)$ submitted a bid $b \notin [0, 1 - p + vp \ln p]$ in the first auction, then he bids in the second auction as if he had submitted $b = 0$ or $b = 1 - p + vp \ln p$, whichever is closer to his actual bid in the first auction.

winner's identity are disclosed. This implies that the equilibrium strategies in the first auction are also the same for both disclosure rules. Thus, one can conclude that disclosing only the winning bid and disclosing only the winner's identity are outcome equivalent. Note, though, that the equivalence result was different when the valuations were negatively correlated. In that case, disclosing the winner's identity was equivalent to disclosing only the losing bid.

4.4 Only The Losing Bid Is Disclosed

Here and in the next subsection, I do not require that $v \leq 1$.

Proposition 11 Item 1: *If* $p < \frac{2v}{1+2v}$, *then* $F : \left[0, 1 + pv - \sqrt{pv\,(2 + pv)}\right] \to [0, 1]$, *where*

$$F\left(b\right) = \frac{p}{1-p}\frac{b}{1-b} + \frac{\sqrt{pv\left(2+pv\right)} - p(1+v)}{1-p}\frac{1}{1-b}. \tag{14}$$

If $\frac{2v}{1+2v} \leq p < 1$, *then* $F : [0, 1-p] \to [0, 1]$, *where*

$$F\left(b\right) = \frac{p}{1-p}\frac{b}{1-b}.$$

Item 2: *The bidders bid as specified in Lemma 8, where* $q_w = q_l = 0$ *if* $b_l > 0$*; otherwise*

$$\begin{aligned} q_w &= \frac{\frac{1}{2}p}{\frac{1}{2}\left(p + (1-p)\,F\left(0\right)\right) + (1-p)\left(1 - F\left(0\right)\right)}, \\ q_l &= \frac{p}{p + (1-p)\,F\left(0\right)}. \end{aligned}$$

The equilibrium payoff of type $(1, 1)$ *is* $\sqrt{pv\left(2+pv\right)}$ *when* $p < \frac{2v}{1+2v}$ *and it is* $p(1 + v)$ *when* $\frac{2v}{1+2v} \leq p < 1$.

Note that $F(0) = 0$ in (14) when $p = \frac{2v}{1+2v}$. Therefore, the first expression for F only applies when $p < \frac{2v}{1+2v}$. In this case, type $(1, 1)$ bids 0 with a strictly positive probability and, as a result, can lose to type $(0, 0)$ opponent. Therefore, the equilibrium is inefficient when $p < \frac{2v}{1+2v}$.

When $\frac{2v}{1+2v} \leq p < 1$, the bidders' payoffs and the seller's revenue in the equilibrium are the same as in the case of no information disclosure. That is, the ex ante payoff of a bidder is given by $p(1-p)(1+v)$ and the seller's revenue is given by

$$R_p^l = (1-p)^2(1+v).$$

When $p < \frac{2v}{1+2v}$, the ex ante equilibrium payoff of a bidder is $(1-p)\sqrt{pv\,(2+pv)}$. The surplus in the first auction will be 0 if a bidder with 0 valuation wins that auction. This happens with a probability of

$$p^2 + 2p(1-p)F(0) \times \frac{1}{2} = p(\sqrt{pv\,(2+pv)} - pv).$$

The surplus in the second auction is 0 if both bidders have 0 valuations, which happens with a probability of p^2. Hence, the expected surplus from the sequential auction is

$$1 - p(\sqrt{pv\,(2+pv)} - pv) + (1-p^2)v = 1 + v - p\sqrt{pv\,(2+pv)}.$$

Finally, the seller's revenue is given by

$$\begin{aligned} R_p^l &= 1 + v - p\sqrt{pv\,(2+pv)} - 2(1-p)\sqrt{pv\,(2+pv)} \\ &= 1 + v - (2-p)\sqrt{pv\,(2+pv)}. \end{aligned}$$

4.5 The Winning and Losing Bids Are Disclosed

It turns out that the equilibrium outcome is equivalent to the one found in the case when only the losing bid is disclosed, again, because the formula for

q_l remains unchanged.

Proposition 12 *The equilibrium strategies are the same as in Proposition 11, except that*

$$q_i = \begin{cases} \frac{p}{p+(1-p)F(0)} & \text{if} \quad b_w = 0, \\ 0 & \text{if} \quad b_l > 0, \end{cases}$$

for $i = w, l$, *and* $q_w = 0$ *and* $q_l = \frac{p}{p+(1-p)F(0)}$ *if* $0 = b_l < b_w$.

Proof. It is easy to verify that the beliefs q_i for $i = w, l$ are consistent with the strategies in the first auction, and that $q_w \leq q_l$ indeed holds. The only difference from Proposition 11 is in the value of q_w, but it does not affect the expected payoffs in the second auction. Therefore, the equilibrium strategies in the first auction are exactly the same as in Proposition 11. ■

4.6 The Ranking of Revenues

Again, given the outcome equivalence between several disclosure rules, it is enough to consider the seller's revenue for the following cases: no information is disclosed, only the winning bid is disclosed, and only the losing bid is disclosed. First of all, one can verify that the seller's revenue under non-disclosure, $(1-p)^2(1+v)$, (weakly) exceeds both R_p^w and R_p^l. Thus, if the valuations are perfectly positively correlated, the best that the seller can do, is not to disclose any information between the first and second auctions. Alternatively, she can sell both objects in a simultaneous auction, in which the bidders either submit separate bids for each object, or they each submit a single bid for the bundle of both objects.

One may also want to compare R_p^w and R_p^l. Here I assume that $v \leq 1$.

Let $\Delta_p = R_p^w - R_p^l$. Then, Δ_p is given by

$$\Delta_p = \begin{cases} (2-p)\left(\sqrt{pv\,(2+pv)} - p(1+v)\right) + 2p\,(1-p)\,v\ln p & \text{if} \quad p < \frac{2v}{1+2v}, \\ 2p\,(1-p)\,v\ln p & \text{if} \quad \frac{2v}{1+2v} \leq p < 1. \end{cases}$$

It immediately follows that $\Delta_p < 0$ when $\frac{2v}{1+2v} \leq p < 1$. $\Delta_p = 0$ is plotted in Figure 3. For the values of p and v to the left of $\Delta_p = 0$ disclosing the winning bid is better, otherwise disclosing the losing bid is better. In particular, if p exceeds $p^* \approx 0.2325$, disclosing the losing bid is better for all values of v. The intuition for the ranking of revenues is simple when $\frac{2v}{1+2v} \leq p < 1$: although the surplus under both disclosure rules is the same, the bidders expect higher payoff when the winning bid is disclosed. On the other hand, if $p < \frac{2v}{1+2v}$, the surplus is not maximized when the losing bid is disclosed. This should work in favour of disclosing the winning bid. However, for many parameter values, this effect is reversed by the fact that the bidders also bid more aggressively and, consequently, expect lower payoffs when the losing bid is disclosed. This is illustrated in Figure 3 by $\delta_p = 0$ where $\delta_p = p\,(1 + v\,(1 - \ln p)) - \sqrt{pv\,(2+pv)}$ is the difference in a bidder's payoff when the winning bid and the losing bid, respectively, are disclosed.

5 Discussion

I have characterized equilibria and compared the corresponding equilibrium outcomes across different disclosure rules when the valuations are perfectly correlated. One may ask how the results will change once we depart from the assumption of perfect correlation. Note that if the valuations are not correlated across items, then the disclosure rule is irrelevant. That is, the no-correlation case seems to act as a dividing line and I expect that the results that I have established for perfect positive or negative correlation, will continue to hold for imperfect correlation as long as the sign of correlation does not change. For example, I expect that disclosing the information about bids will raise seller's revenues as long as the valuations are negatively, even if imperfectly, correlated, but will lower the revenue as long as the valuations are positively, even if imperfectly, correlated.

I have shown for negatively correlated valuations that a more elaborate disclosure rule can lead to even higher revenue for the seller. It would be interesting to characterize what is the optimal disclosure rule in this case. As a first step, one could identify the optimal (static) mechanism (subject to the constraint that both items must be sold) and then check whether or not the sequential first-price auction with the disclosure rule of Section 3.7 replicates this optimal mechanism. Further, besides studying the effects of information disclosure, the model of sequential first-price auction can be used to analyze other issues. Thus, when $v \neq 1$, the items are heterogenous and the order, in which the items are sold, becomes important. One may ask how the optimal order of sales depends on the parameters of the model for a given disclosure rule, or even how the optimal order of sales changes with the disclosure rule. To do that, one first needs to characterize the equilibrium strategies for some of the disclosure rules when $v > 1$. I leave these questions for the future research.

I finish by briefly analysing another issue that has received attention in the literature, namely, dynamics of equilibrium prices. It has been noted in the empirical literature that the prices tend to decline for the items that are sold later in a sequential auction even if all items are identical. (References to this literature can be found, for example, in Section 4.5 and Table 7 of the survey by Ashenfelter and Graddy (2003).) This phenomenon is known as the declining price anomaly. I show next that the expected equilibrium prices decline when the valuations are negatively correlated, thus providing another possible explanation to this phenomenon.

5.1 Dynamics of Equilibrium Prices

I consider the case when the seller only announces the winning bid, which is studied in Section 3.2.

The distribution of the winning bid, b_w, in the first auction is given by

$$(pF_{01}(b_w) + (1-p)F_{10}(b_w))^2 = \left(\frac{pF_{01}(0)}{1-b_w}\right)^2.$$

The expected value of the winning bid in the first auction is

$$E[b_w] = \int_0^{\bar{b}_{10}} b_w d\left(\frac{pF_{01}(0)}{1-b_w}\right)^2 = (1-pF_{01}(0))^2,$$

where I have used that $\bar{b}_{10} = 1 - pF_{01}(0)$. Conditional on b_w, the distribution of the winning bid in the second auction, which is denoted as c_w, is

$$(q_w + (1-q_w)G_w(c_w))(q_l + (1-q_l)G_l(c_w)) = \left(\frac{q_w v}{v-c_w}\right)^2.$$

The expected value of c_w, conditional on b_w, is

$$\int_0^{(1-q_w)v} c_w d\left(\frac{q_w v}{v-c_w}\right)^2 = (1-q_w)^2 v.$$

Finally, we integrate out b_w to obtain the expected value of c_w,

$$E[c_w] = (pF_{01}(0))^2 v + \int_0^{\bar{b}} (1-q_w)^2 v d\left(\frac{pF_{01}(0)}{1-b_w}\right)^2,$$

where the first term captures the fact that $c_w = v$ if both bidders are of type $(0,1)$ and both bid 0 in the first auction. Substituting the expression for q_w, one arrives at

$$E[c_w] = (pF_{01}(0))^2 \left(v + \frac{1}{v} \int_0^{\bar{b}} \frac{2(v + \ln(1-b_w))^2}{(1-b_w)^3} db_w \right).$$

To simplify the calculations, I now assume that $v = 1$ and $p = 0.5$. With these parameter values, the expected equilibrium price in both auctions is 0.25 if the seller does not disclose any information between the two auctions. I now show that the prices are declining if the seller discloses the winning bid. For $v = 1$ and $p = 0.5$, $F_{01}(0) = \frac{1}{e-1}$ and $\bar{b} = 1 - e^{-1}$. Using these values, one finds that

$$E[b_w] = \left(\frac{2e-3}{2e-2} \right)^2 \approx 0.5027$$

and

$$E[c_w] = \frac{e^2 - 3}{8\,(e-1)^2} \approx 0.1858.$$

To summarize,

Lemma 13 *Assume that* $v = 1$, $p_{00} = p_{11} = 0$, *and* $p_{01} = p_{10} = 0.5$. *Then the expected equilibrium prices are declining,* $E[b_w] > E[c_w]$, *if the seller discloses the winning bid of the first auction before the start of the second auction.*

Appendix

Proof of Proposition 3. First of all, one can verify that the bidders' beliefs at the start of the second auction, as described by the first expression for each q_i for $i = w, l$, are consistent with the strategies of the first auction. (Although, bidder l additionally knows his losing bid, it does not provide any additional information about the opponent's type.) Further, the density functions for $0 \leq b \leq \bar{b}$ are given by

$$\begin{aligned} f_{01}(b) &= F_{01}(0) \frac{v + \ln(1-b)}{v(1-b)^2}, \\ f_{10}(b) &= F_{01}(0) \frac{p}{1-p} \frac{-\ln(1-b)}{v(1-b)^2}. \end{aligned}$$

One can verify that $f_{01}(b) > 0$ and $f_{10}(b) > 0$ for all $0 < b < \bar{b}$ as required. Using these expressions for $f_{01}(b)$ and $f_{10}(b)$, one can simplify the expressions for q_w and q_l when $0 \leq b_w \leq \bar{b}$:

$$\begin{aligned} q_w &= -\frac{\ln(1-b_w)}{v}, \\ q_l &= -\frac{b_w + \ln(1-b_w)}{v}. \end{aligned}$$

Hence, $q_l \leq q_w$ clearly holds, therefore one can apply Lemma 1. Finally, one can also verify that $f_{01}(1-e^{-v}) = 0$. This is important because it ensures that there is no discrete jump in q_w at $b_w = \bar{b} = 1 - e^{-v}$ when $pv < 1 - e^{-v}$. Consequently, $q_w = 1$ for all $b \geq \bar{b}$ when $pv < 1 - e^{-v}$.

Next, we consider the bidders' behaviour in the first auction. Consider type $(0,1)$. If he bids in the interval $[0, \bar{b}]$, his joint payoff from both auctions is[18]

$$\begin{aligned}
\pi_{01} &= \left(pF_{01}(b) + (1-p)F_{10}(b)\right) \times (-b) \\
&\quad + \left(pF_{01}(b) + (1-p)F_{10}(b)\right) \times \frac{(1-p)f_{10}(b)}{pf_{01}(b) + (1-p)f_{10}(b)} \times v \\
&\quad + \left(p(1-F_{01}(b)) + (1-p)(1-F_{10}(b))\right) \\
&\quad \times \int_{b}^{\bar{b}_{10}} v \times \frac{(1-p)f_{10}(s)}{pf_{01}(s) + (1-p)f_{10}(s)} \times \frac{pf_{01}(s) + (1-p)f_{10}(s)}{p(1-F_{01}(b)) + (1-p)(1-F_{10}(b))} ds,
\end{aligned}$$

where the first term is the bidder's payoff from the first auction, the second and third terms are his payoff from the second auction if he has, respectively, won and lost the first auction,[19] and $\bar{b}_{10} = \bar{b}$ if $pv \geq 1-e^{-v}$, and $\bar{b}_{10} = 1-\frac{pv}{e^v-1}$ otherwise. If we simplify the above expression, we obtain that

$$\begin{aligned}
\pi_{01} &= -\left(pF_{01}(b) + (1-p)F_{10}(b)\right)(b + \ln(1-b)) + (1-p)(1-F_{10}(b))v \\
&= (1-p)v,
\end{aligned}$$

where the last equality follows from substituting the expressions for $F_{01}(b)$ and $F_{10}(b)$.

If he bids above $\bar{b}$, the expression for the payoff depends on whether $pv < 1-e^{-v}$ or $pv \geq 1-e^{-v}$. If $pv < 1-e^{-v}$, then $q_w = 1$ and for $b \in \left(\bar{b}, 1-\frac{pv}{e^v-1}\right]$, the payoff is

$$(p + (1-p)F_{10}(b))(v-b) + (1-p)(1-F_{10}(b))v = v\left(1 - \frac{p}{e^v-1}\frac{b}{1-b}\right),$$

where I have used (4). Because $b > \bar{b} = 1-e^{-v}$, this payoff is less than $(1-p)v$. For $b > 1-\frac{pv}{e^v-1}$, the payoff is

$$v - b < v - 1 + \frac{pv}{e^v-1} < v - 1 + e^{-v} < (1-p)v.$$

[18] If $b = 0$, we need to consider ties. However, the first two terms in the payoff expression will be zero for any tie-breaking rule.

[19] Recall that the payoff of type $(0,1)$ from the second auction is given by $q_w v$, where q_w depends on the winning bid of the first auction.

If $pv \geq 1 - e^{-v}$, then for $b > \bar{b}$ the payoff is

$$-b + \frac{(1-p) f_{10}\left(\bar{b}\right)}{p f_{01}\left(\bar{b}\right) + (1-p) f_{10}\left(\bar{b}\right)} v = -b - \ln\left(1 - \bar{b}\right) < -\bar{b} - \ln\left(1 - \bar{b}\right) = (1-p)v.$$

It follows that the payoff of type $(0,1)$ is maximized for $b \in \left[0, \bar{b}\right]$. Thus, it is indeed optimal for this type to randomize on this interval.

Consider type $(1,0)$. If he bids in the interval $\left(0, \bar{b}\right]$, his expected payoff is

$$\pi_{10} = \left(pF_{01}(b) + (1-p) F_{10}(b)\right)(1-b) = pF_{01}(0),$$

where the last equality follows from substituting the expressions for $F_{01}(b)$ and $F_{10}(b)$. If $pv < 1 - e^{-v}$, then for any $b \in \left(\bar{b}, 1 - \frac{pv}{e^{v}-1}\right]$, the expected payoff is

$$\left(p + (1-p) F_{10}(b)\right)(1-b) = pF_{01}(0),$$

where I have used (4). If he bids 0, then his expected payoff is $\frac{1}{2} pF_{01}(0)$, given the tie-breaking rule. If he bids above $\bar{b}_{10}$, his expected payoff is $1 - b < 1 - \bar{b}_{10} = pF_{01}(0)$. Thus, this type is indifferent between all bids $\left(0, \bar{b}_{10}\right]$ and, hence, is indeed willing to randomize according to $F_{10}(b)$. ■

Proof of Proposition 5. First, the bidders' beliefs at the start of the second auction, as described by q_w and q_l, are derived from the strategies of the first auction using Bayes' rule whenever possible. Second, because $q_l \leq q_w$ holds, one can apply Lemma 1 to the second auction.

We now turn to the first auction. First, however, note that given the specified strategies, ties will arise with zero probability. (In particular, $\lim_{b \to ((1-p)v)^{+}} F_{01}(b) = 0$.) Consider type $(0,1)$. If he bids in the interval

$\left[(1-p)v, -\ln(1-p)\frac{1-p}{p}v\right]$, his expected payoff from both auctions is

$$\begin{aligned}
\pi_{01} &= pF_{01}(b)(-b) + (1-p+p(1-F_{01}(b)))\frac{1-p}{1-p+p(1-F_{01}(b))}v \\
&\quad + pF_{01}(b)\int_{(1-p)v}^{b}\frac{1-p}{1-p+p(1-F_{01}(s))}v\frac{f_{01}(s)}{F_{01}(b)}ds \\
&= pF_{01}(b)(-b) + (1-p)v - (1-p)v\int_{(1-p)v}^{b}\frac{d(1-pF_{01}(s))}{1-pF_{01}(s)} \\
&= pF_{01}(b)(-b) + (1-p)v - (1-p)\ln(1-pF_{01}(b))v \\
&= (1-p)v,
\end{aligned}$$

where the last equality follows from the definition of $F_{01}(b)$ in (5). Thus, type $(0,1)$ is indifferent between all bids in the interval $\left[(1-p)v, -\ln(1-p)\frac{1-p}{p}v\right]$. If this type bids below $(1-p)v$, he expects the same payoff as when bidding $(1-p)v$. If he bids above $-\ln(1-p)\frac{1-p}{p}v$, he expects a payoff of

$$\begin{aligned}
& (p+(1-p)F_{10}(b))(-b) + (1-p)v - (1-p)\ln(1-p)v \\
= \; & (p+(1-p)\ln(1-p)v)\frac{-b}{1-b} + (1-p)(1-\ln(1-p))v,
\end{aligned}$$

which is decreasing in b because $p+(1-p)\ln(1-p)v > 0$. It follows that it is indeed optimal for type $(0,1)$ to randomize according to $F_{01}(b)$ in the first auction.

Consider type $(1,0)$. If he bids in the interval $\left[-\ln(1-p)\frac{1-p}{p}v, (1-p)(1-\ln(1-p)v)\right]$, his expected payoff is

$$\begin{aligned}
\pi_{10} &= (p+(1-p)F_{10}(b))(1-b) \\
&= \left(p+(1-p)\frac{p}{1-p}\frac{b+\ln(1-p)\frac{1-p}{p}v}{1-b}\right)(1-b) \\
&= p+(1-p)\ln(1-p)v.
\end{aligned}$$

If he bids above $(1-p)(1-\ln(1-p)v)$, he clearly expects less. If he bids below $-\ln(1-p)\frac{1-p}{p}$, his payoff is

$$pF_{01}(b)(1-b) = pF_{01}(b) + (1-p)\ln(1-pF_{01}(b))v,$$

where I have used (5). One can verify that the last expression is increasing in b for $v \leq 1$. Thus, it follows that it is indeed optimal for type $(1,0)$ to randomize according to $F_{10}(b)$ in the first auction. ■

Proof of Proposition 6. We solve for the equilibrium strategies backwards, starting with the second auction. Given the specified strategies, it is clear that type $(1,0)$ finds it optimal to bid 0 for any beliefs about the type of the opponent. It remains to check the optimality of type $(0,1)$ strategies. At the beginning of second auction, bidder w of type $(0,1)$ believes that the opponent is of type $(1,0)$ with probability $q_l = 0$. His expected payoff at the start of the second auction is

$$H_l(c)(v-c) = q_w v$$

for all $c \in [0,(1-q_w)v]$ and it is $v-c < q_w v$ for $c > (1-q_w)v$. Thus, this bidder is willing to bid c_w as long as $c_w \leq (1-q_w)v = \frac{p(1-F_{01}(b_l))}{1-p+p(1-F_{01}(b_l))}v = c_l$ where the first equality follows from (6) and the second equality is simply a definition ($c_l \equiv c(b_l)$). Because $c(b)$ is decreasing in b and $b_w \geq b_l$, it is indeed true that $c_w \leq c_l$.

On the other hand, bidder l of type $(0,1)$ believes that the opponent is of type $(1,0)$ with probability q_w, which is given in (6). Furthermore, conditional on his first auction losing bid b_l and conditional on the opponent being of type $(0,1)$, bidder l believes that the opponent's bid in the first auction exceeds b with probability $\frac{1-F_{01}(b)}{1-F_{01}(b_l)}$, or equivalently that the opponent's bid in the second auction is less than c with probability $\frac{H_w(c)}{H_w(c_l)}$ where $c_l = (1 - q_w)v$. Therefore, the expected payoff of this type at the start of the second auction is

$$\left(q_w + (1 - q_w) \frac{H_w(c)}{H_w(c_l)} \right) (v - c).$$

Substituting the expressions for q_w and $H_w(c)$, we obtain that

$$\left(\frac{1 - p + p(1 - F_{01}(b_l)) \frac{H_w(c)}{H_w(c_l)}}{1 - p + p(1 - F_{01}(b_l))} \right) (v - c) = \frac{1 - p}{1 - p + p(1 - F_{01}(b_l))} v = q_w v$$

for all $c \in [0, (1 - q_w)v]$. If he bids above $(1 - q_w)v$, his payoff is clearly less than $q_w v$. Because this type of bidder is indifferent between all bids in the interval $[0, (1 - q_w)v]$, it is indeed optimal to randomize according to $H_l(c)$.

Hence, type $(0, 1)$ expects a payoff of $q_w v$, which depends on b_l, in the second auction whether or not he has won the first auction. It immediately follows that the equilibrium strategies in the first auction are exactly the same as the ones in Proposition 5. ■

Proof of Proposition 7. First, the bidders' beliefs at the start of the second auction, as described by q_w and q_l, are derived from the strategies of the first auction using Bayes' rule whenever possible. In particular, one cannot apply Bayes' rule to determine q_w if $b_w \in (0, k)$ is announced, or q_l if $b_l \in (0, k)$ is announced, because no bids are submitted in this region in the equilibrium. It is assumed in this case that the deviator is type $(0, 1)$ bidder. Second, because $q_l \leq q_w$ holds, one can apply Lemma 1 to the second auction.

We now turn to the first auction. Consider type $(0, 1)$. Suppose he bids 0. Then the announcement by the seller will be either that the winning bid is 0, in which case it will be common knowledge that $q_l = q_w = 0$ and the bidders will expect a payoff of 0 in the second auction, or that the losing bid is 0, in which case type $(0,1)$ bidder expects a payoff of $\frac{1-p}{1-p+p(1-F_{01}(0))}$ from the second action. Hence, the expected payoff is

$$pF_{01}(0)\times 0+(1-p+p(1-F_{01}(0))\times\frac{1-p}{1-p+p(1-F_{01}(0))}=1-p.$$

If he bids in the interval $\left[k,\bar{b}_{01}\right]$, his expected payoff from both auctions is

$$\begin{aligned}
& pF_{01}(b)(-b)+pF_{01}(0)\frac{1-p}{1-p+p(1-F_{01}(0))} \\
& +p(F_{01}(b)-F_{01}(k))\int_{k}^{b}\frac{1-p}{1-p+p(1-F_{01}(s))}\frac{f_{01}(s)}{F_{01}(b)-F_{01}(k)}ds \\
& +(1-p+p(1-F_{01}(b)))\frac{1-p}{1-p+p(1-F_{01}(b))} \\
= \; & pF_{01}(b)(-b)+pF_{01}(0)k \\
& -(1-p)(\ln(1-pF_{01}(b))-\ln(1-pF_{01}(k)))+1-p \\
= \; & (1-p),
\end{aligned}$$

where I have used that $\frac{1-p}{1-p+p(1-F_{01}(0))}=k$, and the last equality follows from the definition of $F_{01}(b)$. Thus, type $(0,1)$ is indifferent between all bids in $\{0\}\cup\left[k,\bar{b}_{01}\right]$, and he expects a payoff $\pi_{01}=1-p$.

If this type bids in $(0,k)$, he expects a payoff of

$$pF_{01}(b)(-b+0)+(1-p+p(1-F_{01}(b))\times\frac{1-p}{1-p+p(1-F_{01}(b))}<1-p.$$

If he bids above $\bar{b}_{01}$, he expects a payoff of

$$
\begin{aligned}
& (p+(1-p)F_{10}(b))(-b)+pF_{01}(0)k \\
& -(1-p)(\ln(1-p)-\ln(1-pF_{01}(k)))+1-p \\
= \; & p\frac{1-\bar{b}_{01}}{1-b}(-b)+p\bar{b}_{01}+1-p \\
= \; & p\frac{\bar{b}_{01}-b}{1-b}+1-p<1-p,
\end{aligned}
$$

where I have used the definition of $F_{10}(b)$ and the fact that

$$
p\left(-\bar{b}_{01}\right)+pF_{01}(0)k=(1-p)(\ln(1-p)-\ln(1-pF_{01}(k))). \tag{15}
$$

It follows that it is indeed optimal for type $(0,1)$ to randomize according to $F_{01}(b)$ in the first auction.

Consider now type $(1,0)$. If he bids in the interval $[\bar{b}_{01},\bar{b}_{10}]$, his expected payoff is

$$
\pi_{10}=(p+(1-p)F_{10}(b))(1-b)=p(1-\bar{b}_{01}).
$$

He has no incentives to bid above $\bar{b}_{10}$. If he bids in the interval $\left[k,\bar{b}_{01}\right]$, his expected payoff is

$$
\begin{aligned}
& pF_{01}(b)(1-b) \\
= \; & pF_{01}(b)-pF_{01}(k)k+(1-p)(\ln(1-pF_{01}(b))-\ln(1-pF_{01}(k))),
\end{aligned}
$$

which is increasing in b.

Finally, because the probability of winning the auction is the same for all bids in $(0,k)$, it is enough to consider type $(1,0)$ bidding “just above 0”. This deviation is also better than bidding exactly 0 as it avoids possible ties with the opponent. The payoff from bidding just above 0 is $pF_{01}(0)$. It remains to check when $pF_{01}(0) \leq p(1-\bar{b}_{01})$ holds. Using $\bar{b}_{01} = \frac{k+p-1-(1-p)\ln(k)}{p}$, which follows from (15), and $F_{01}(0) = \frac{k+p-1}{kp}$, we find that type $(1,0)$ will not want to deviate if $1-p \geq \frac{k^2}{1+k\ln(k)}$, which is indeed satisfied. Thus, it is optimal for type $(1,0)$ to randomize according to $F_{10}(b)$ in the first auction. ■

Proof of Proposition 9. First, the beliefs at the start of the second auction round are consistent with the first round strategies. Second, $q_w \leq q_l$ holds. Third, the lowest bid that type $(1,1)$ submits in the first auction is 0, therefore it is indeed optimal for type $(0,0)$ to bid 0 in that auction.

The expected payoff of type $(1,1)$ bidder at the start of the first auction is

$$(p+(1-p)F(b))(1-b)+(p+(1-p)F(b))\frac{pv}{p+(1-p)F(b)}$$
$$+(1-p)(1-F(b))\int_b^{\bar{b}}\frac{pv}{p+(1-p)F(s)}\frac{f(s)}{1-F(b)}ds$$

where the first term is the bidder's payoff from the first auction, the second and third terms are his payoff from the second auction if he has, respectively, won and lost the first auction,[20] and $\bar{b} \equiv 1-p+vp\ln p$. After simplification, we obtain

$$(p+(1-p)F(b))(1-b)+vp-vp\ln(p+(1-p)F(b)).$$

Substituting the expression for $F(b)$ from (10) implies that the payoff is equal to $p(1+v(1-\ln p))$ for all $b \in [0,\bar{b}]$. Clearly, the payoff is strictly lower if the bidder bids above $\bar{b}$. Hence, a bidder with the valuation profile $(1,1)$ is indeed willing to randomize according to $F(b)$ in the first auction. This completes the proof. ■

[20]Recall that the payoff of type $(1,1)$ from the second auction is given by $q_l v$, where q_l depends on the winning bid of the first auction according to (11).

Proof of Proposition 10. It is trivial to see that it is optimal for type $(0,0)$ to bid 0 in both auctions, given the strategy of the opponent. Consider type $(1,1)$. We start with the second auction. Prior to the second auction, bidder l (of type $(1,1)$) believes that the opponent's type is $(0,0)$ with probability $q_w = 0$, while bidder w believes that his opponent's type is $(0,0)$ with probability q_l, which is given in (11).

The expected payoff of bidder l, conditional on c_w, is

$$H_w(c)(v-c) = v - c_w$$

for $c \leq c_w$ and it is $v - c < v - c_w$ for any $c > c_w$. Thus, even if bidder l does not know what c_w is, he knows that his payoff is maximized at $c = c_l$ (because $c_l \leq c_w$).

The expected payoff of bidder w is

$$\left(q_l + (1-q_l)\frac{H_l(c)}{H_l(c_w)}\right)(v-c).$$

Substituting the expressions for q_l and $H_l(c)$ in the payoff function and noting that $F(b_w) = H_l(c_w)$, we obtain that the payoff is $v - c_w$ for $c \leq c_w$, but the payoff is $v - c < v - c_w$ for $c > c_w$. Thus, bidder w is indeed willing to randomize according to $H_w(c)$.

Note that from (11) and (13), $c_w = (1-q_l)v$. It follows that type $(1,1)$ expects a payoff of $q_l v$ in the second auction whether he wins or loses the first auction. From this and the fact that the relationship between q_l and b_w is the same as when the winning bid is disclosed, it immediately follows that the equilibrium strategy of type $(1,1)$ in the first auction is exactly the same as the one given in Proposition 9. This completes the proof. ■

Proof of Proposition 11. The beliefs prior to the second auction are indeed consistent with the bidding strategies of the first auction; $q_w \leq q_l$ holds; type $(0,0)$ bidder finds it optimal to bid 0 in the first auction.

Consider type $(1,1)$ in the first auction when $p < \frac{2v}{1+2v}$. If he bids $b > 0$, then his expected payoff is

$$\begin{aligned}&(p + (1-p)\,F\,(b))\,(1-b) + (p + (1-p)\,F\,(0)) \times \frac{pv}{p + (1-p)\,F\,(0)}\\&+ (1-p)\,(1 - F\,(0)) \times 0.\end{aligned}$$

If he bids $b = 0$, then the payoff is

$$(p + (1-p)\,F\,(0)) \times \frac{1}{2} \times 1 + \frac{pv}{p + (1-p)\,F\,(0)}.$$

Substituting the expression for $F\,(b)$, it follows that the expected payoff both from bidding $b > 0$ and from bidding $b = 0$ is the same and equal to $\sqrt{pv\,(2+pv)}$. Obviously, the bidder has no incentives to bid above $\bar{b} = 1 + pv - \sqrt{pv\,(2+pv)}$ as it is dominated by bidding $\bar{b}$.

When $\frac{2v}{1+2v} \leq p < 1$, type $(1,1)$ expects

$$(p + (1-p)\,F\,(b))\,(1-b) + p \times v + (1-p) \times 0 = p(1+v)$$

for $0 < b \leq 1-p$. If he bids $b = 0$, his expected payoff is

$$p \times \frac{1}{2} \times 1 + v,$$

which is (weakly) less than $p(1+v)$ when $\frac{2v}{1+2v} \leq p < 1$. He has no incentives to bid above $1-p$ either. We conclude that F is indeed the equilibrium strategy of type $(1,1)$ bidder in the first auction. ■

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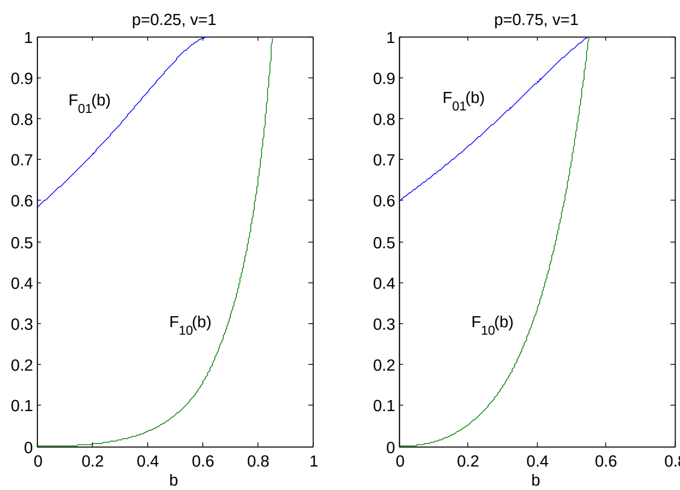

Figure 1: Equilibrium strategies

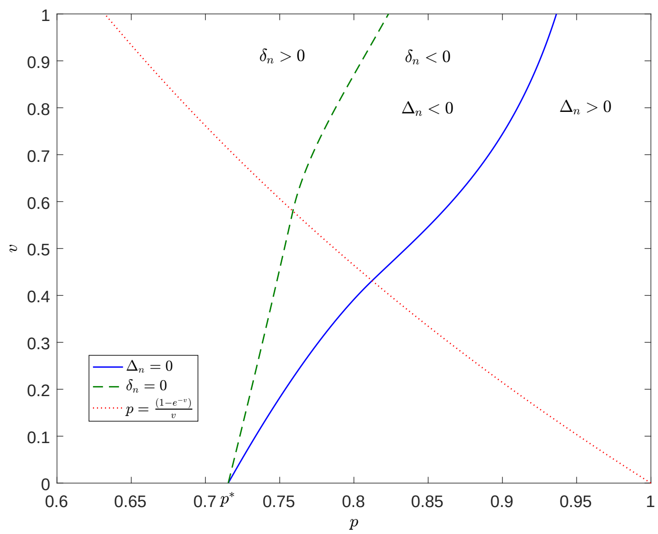

Figure 2: The ranking of revenues

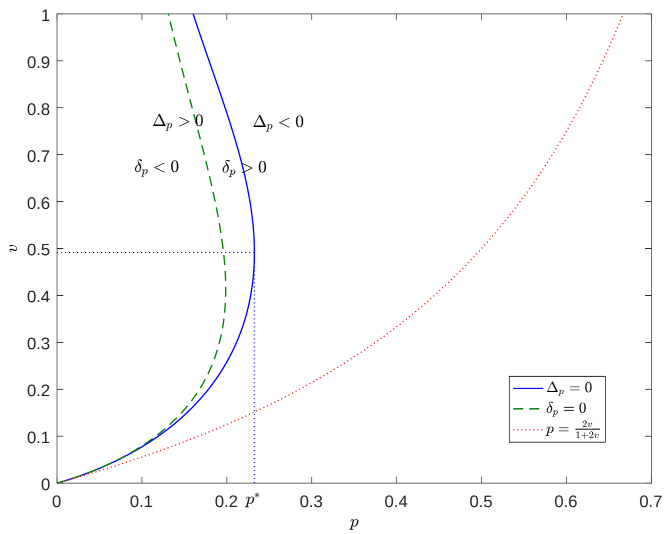

Figure 3: The ranking of revenues