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From Sunspots to Black Holes: Singular dynamics in macroeconomic models *

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Abstract

We present conditions for the emergence of singularities in DGE models. We distinguish between slow-fast and impasse singularity types, review geometrical methods to deal with both types of singularity and apply them to DGE dynamics. We find that impasse singularities can generate new types of DGE dynamics, in particular temporary determinacy/indeterminacy. We illustrate the different nature of the two types of singularities and apply our results to two simple models: the Benhabib and Farmer (1994) model and one with a cyclical fiscal policy rule.

KEYWORDS: slow-fast singularities, impasse singularities, macroeconomic dynamics, temporary indeterminacy

JEL CODES: C62, D43, E32

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1 Introduction

Jean-Michel Grandmont has been interested in the rigorous mathematical analysis of non-linear dynamic systems in economic applications throughout his long career. In particular, he has often focussed on the less standard aspects of dynamics revealed by the mathematics, such as chaos and sunspots - Grandmont (1985), Grandmont (1998), Grandmont (2008), and Grandmont, Pintus, and Vilder (1998) provide us some fine examples of his contribution to our understanding on how endogenous fluctuations may emerge in competitive economies. This line of research has been associated with the study of models with multiple equilibria, local indeterminacy, and sunspot equilibria, which have became part of mainstream macroeconomics in the last three decades.

In this article we explore something that Grandmont did not get around to studying: the possibility of singular dynamics in economic models due to infinite eigenvalues. We believe that infinite eigenvalues are more than a simple curiosity. With finite eigenvalues the resulting non-singular dynamics display the property that the dimension of the stable manifold is always the same, so that there is either permanent determinacy or permanent indeterminacy of equilibrium paths. However, we believe that this is unrealistic: casual empiricism suggests that economies can pass through periods when they are more volatile (corresponding to indeterminacy) and periods when they are less volatile (corresponding to determinate dynamics). Of course, there might be many explanations of this, but the presence of infinite eigenvalues opens up the possibility that the dimension of the stable manifold can change at a point in time resulting in a "natural" and endogenous change in the determinacy of the equilibrium path in real (finite) time. The determinacy (indeterminacy) of an economy can thus be a temporary phenomenon and the economy can switch between determinate and indeterminate dynamics along its equilibrium path without the intervention of changes in underlying parameters.

We provide a general analysis of a generic dynamic general equilibrium (DGE) model and show that two different types of singularity can arise. One is the case where a parameter varies and at a particular value renders the eigenvalues infinite everywhere. This is a *singular perturbation* and calls for a method of solution dealing with *slow-fast* systems. The second is a case of *impasses* for which there is a one-dimensional impasse set where eigenvalues become locally infinite. This creates a barrier which can only be crossed at a specific point or isolated points at which particular properties are satisfied. This is the case where eigenvalues change sign from plus infinity to minus infinity (or the other way around). In this case, if an equilibrium path crosses this barrier, it does so at a particular point in time. The determinacy properties will switch at that point in time to reflect the change in sign of the eigenvalue. For example, if we have a path that approaches the barrier with two negative eigenvalues, one of which goes to infinity, this will exert a strong pull towards a point on the barrier, which will be reached in finite time. If that point on the barrier satisfies certain conditions (see Proposition 4 in section 2.4), the path will emerge at the other side, but with one positive eigenvalue and one negative at which point it can continue to the steady-state. Hence, there is a period of temporary indeterminacy: prior to reaching the barrier, the point acts as a sink and the dynamics are indeterminate. Once the barrier is passed, the dynamics become determinate with a saddle path. In effect, impasses can give rise to a change in the dimension of the stable manifold along the equilibrium path. In this chapter we characterize the conditions required for such a crossing to be possible.

In section 3 we are able to provide two generic economic examples of DGE models that can display singular perturbations and impasses. One is the model of Benhabib and Farmer (1994) in which they noted the possibility that "one root passes through minus infinity and re-emerges as a positive real root." As we demonstrate, this case corresponds to a perturbation singularity and has to be analyzed as a slow-fast dynamic system. The other case is a Ramsey DGE model with endogenous labor in which there is a cyclical fiscal policy rule and distortionary taxation, where we show that not only are impasses possible, but also that they can display the crossing behavior which gives rise to temporary indeterminacy.

Whilst physics and engineering have explored the implications of singularities for some time (e.g. in the study of Black holes), economists have tended to look the other way. There are a few exceptions: Barnett and He (2010) and He and Barnett (2006) are, to our knowledge, the only papers dealing with the importance of singularities in economic models (in their case associated to the introduction of feedback policy rules). Singularities play a large role in other scientific fields and we believe that we also need to understand their potential role in macroeconomic dynamics.

The aim of this article is to present some results on the geometrical properties of local dynamics in a neighborhood of a singularity and to apply them to simple DGE structures. In section 2 we provide an intuition for their existence in a general DGE macro model with endogenous employment. Section 3 provides one example for each type of singularity by applying the methods presented in section 2.

2 A general DGE model with singularities

2.1 The general structure of DGE models

Several DGE models, extending the Ramsey model, feature a semi-explicit differential-algebraic equations (DAE) system in three variables - capital (K), consumption (C), and labor (L):

$$\begin{aligned} \dot{K} &= y(K, L, \varphi) - C \ , & (1) \\ \dot{C} &= C(r(K, L, \varphi) - \rho)\theta(C, L) \ , & (2) \\ 0 &= u_L(C, L) + u_C(C, L)w(K, L) \equiv v(K, C, L) \ , & (3) \end{aligned}$$

together with the conditions

$$\begin{aligned} K(0) &= K_0 \ , & (4) \\ 0 &= \lim_{t \to \infty} u_C(C(t), L(t))K(t)e^{-\rho t} \ , & (5) \end{aligned}$$

where $y(\cdot)$ is an aggregate (net) production function, $r(\cdot)$ is the aggregate (net) return-on-capital function, $\theta(C, L) \equiv -u_C/(Cu_{CC}) > 0$ is the elasticity of intertemporal substitution in consumption that corresponds to the felicity (or utility-flow) function $u(C, L)$ [1], $\rho > 0$ is the discount rate, and $w(\cdot)$ is the aggregate (net) wage-rate function. The parameter vector, including ρ, is generically denoted by φ.

Equation (1) is the instantaneous budget constraint, equation (2) is the Euler equation, and equation (3) is the arbitrage condition between consumption and labor supply (the leisure-consumption trade off). Equations (4) is the initial condition for the stock of capital, where K_0 is a known positive number and (5) is the transversality condition.

Definition 1 (DGE path). *A DGE path is a function $(K(t), C(t), L(t))$ mapping **all** $t \in [0, \infty)$ into a subset of $\mathbb{R}^3_{++}$ which is a solution to the DAE system (1)-(3) such that the initial and the transversality conditions (4)-(5) hold.*

A necessary condition for a solution to (1)-(3) to be a DGE path is that it exists and it is positive *for all* $t \in [0, \infty)$. Solutions of system (1)-(3) that only exist for a finite interval $t \in [0, t_s)$, with t_s finite, cannot be DGE paths.

[1] We use the notation for derivatives $f_x \equiv \frac{\partial y}{\partial x}$ and $f_{xy} \equiv \frac{\partial^2 y}{\partial x \partial y}$.

A steady-state of system (1)-(3) is a point $(\bar{K}, \bar{C}, \bar{L}) \in \mathbb{R}^3_{++}$ such that $\dot{K} = \dot{C} = 0$. steady-states are fixed points of the non-linear equation system

$$\begin{cases} y(K\,L, \varphi) = C \ , \\ r(K, L, \varphi) = \rho \ , \\ v(K, C, L) = 0 \ . \end{cases} \tag{6}$$

A *stationary DGE* is a DGE path such that the variables are permanently at their steady-state levels: $(K(t), C(t), L(t)) = (\bar{K}, \bar{C}, \bar{L})$, for all $t \in [0, \infty)$. It can only exist if $K_0 = \bar{K}$.

An *asymptotic-stationary DGE path* is a DGE path that converges asymptotically to a steady-state, i.e. $\lim_{t\to\infty}(K(t), C(t), L(t)) = (\bar{K}, \bar{C}, \bar{L})$.

From now on, we restrict the analysis to stationary and asymptotic stationary DGE paths, by introducing the following assumption [2]:

Assumption 1. *There is at least one steady-state for system* (1)-(3).

The stable manifold associated to steady-state $(\bar{K}, \bar{C}, \bar{L})$, $\mathcal{W}^s(\bar{K}, \bar{C}, \bar{L})$, is defined as the set of initial points such that the DGE path is asymptotically stationary:

$$\mathcal{W}^s(\bar{K}, \bar{C}, \bar{L}) \equiv \{(K, C, L) \in \mathbb{R}^3_{++} : \lim_{t\to\infty}(K(t), C(t), L(t)) = \big(\bar{K}, \bar{C}, \bar{L}\big)\} \ .$$

Considering that both types of DGE paths (asymptotic-stationary and stationary) satisfy the transversality condition, their existence and uniqueness (or multiplicity) can be assessed by the characteristics of the stable manifolds associated to a steady state $(\bar{K}, \bar{C}, \bar{L})$: (a) If the stable manifold is empty and $K_0 = \bar{K}$, then a DGE path exists and it is stationary; (b) If the stable manifold is non-empty and K_0 belongs to it, then a DGE exists and it is asymptotic-stationary.

Asymptotic-stationary DGE paths can be classified further according to their degree of determinacy (or multiplicity). For this purpose we need to define the local stable manifold ($\mathcal{W}^s_{loc}$) associated to a steady-state $(\bar{K}, \bar{C}, \bar{L})$ as the set of points, belonging to a vicinity of $(\bar{K}, \bar{C}, \bar{L})$, that asymptotically converge to that steady-state :

$$\mathcal{W}^s_{loc}(\bar{K}, \bar{C}, \bar{L}) \equiv \{(K, C, L) \in N : \lim_{t\to\infty}(K(t), C(t), L(t)) = \big(\bar{K}, \bar{C}, \bar{L}\big)\} \ .$$

[2]For simplicity we exclude the existence of periodic solutions, but the analysis can easily be extended to that case.

where N is a neighborhood containing the steady-state such that the Euclidean distance $\|(K, C, L) - (\bar{K}, \bar{C}, \bar{L})\| < \delta$ for a small number $\delta > 0$.

Definition 2 (Asymptotic- determinate and -indeterminate DGE paths)**.** *A DGE path is asymptotic-determinate if the local stable manifold $\mathcal{W}^s_{loc}(\bar{K}, \bar{C}, \bar{L})$ is one-dimensional. A DGE path is asymptotic-indeterminate if the local stable manifold $\mathcal{W}^s_{loc}(\bar{K}, \bar{C}, \bar{L})$ is two-dimensional.*

This means that, given an initial point for K (K_0) sufficiently close to a steady-state, we say there is determinacy if a single DGE path converges to that steady-state and we say there is indeterminacy if an infinite number of paths converge to it.

In section 3 we will show that one possible consequence of the existence of singularities in system (1)-(3) is that the determinacy properties of DGE paths can change along the transition path. Therefore we distinguish further:

Definition 3 (Temporary-determinate and temporary-indeterminate DGE paths)**.** *A DGE path is temporary-(in)determinate if, for a finite t, it belongs to a subset of the stable manifold $\mathcal{W}^s(\bar{K}, \bar{C}, \bar{L})$ that is locally one-dimensional (two-dimensional).*

Definition 4 (Permanent-determinate and permanent-indeterminate DGE paths)**.** *A DGE path displays permanent determinacy (indeterminacy) if the stable manifold is one-dimensional (two-dimensional) for all $t \in [0, \infty)$.*

Two cases are possible in the presence of singularities. First, if the stable manifold has the same degree of determinacy *globally*, then we have permanent determinacy or indeterminacy, depending on the dimension of the local stable manifold. This is the case with regular models, i.e. models without singularities. Second, if the dimension of the stable manifold for points sufficiently far away from the steady-state is *different* from that of the local stable manifold, then we have *temporary* determinacy or indeterminacy or both. In models with particular types of singularities, knowing the dimension of the local stable manifold at the steady-state does not allow us to characterize the determinacy properties of DGE paths sufficiently far away from the steady-state.

Independently from generating a new type of indeterminacy, the existence of singularities can also confine the existence of DGE paths to a subset of the domain of (K, C, L).

2.2 DGE paths in the presence of singularities

Assume that consumption and leisure are substitutes, so that $v(\cdot)$ is monotonic in C. Due to the existence of externalities or any other type of distortion, let $v(\cdot)$ be non-monotonic in L. This property prevents us from eliminating L in equation (3). Instead, by using the implicit-function theorem, we solve equation (3) for C as a function of the other variables: $C = c(K, L)$.

Differentiating $c(K(t), L(t))$ with respect to time and using equations (1) and (2) allows us to obtain a reduced constrained ordinary differential equation (ODE) system in (K, L):

$$\begin{aligned} \dot{K} &= s(K, L, \varphi) \ , \\ c_L(K, L, \varphi)\, \dot{L} &= c(K, L, \varphi)\, (R(K, L, \varphi) - \rho)\, \theta(K, L, \varphi) \ , \end{aligned} \tag{7}$$

where $s(K, L, \varphi) \equiv y(K, L, \varphi) - c(K, L, \varphi)$ is a savings function and

$$R(K, L, \varphi) \equiv r(K, L, \varphi) - \frac{c_K(K, L, \varphi)}{c(K, L, \varphi)\theta(K, L, \varphi)}\ s(K, L, \varphi) \ ,$$

is a modified return-on-capital function. Henceforth, we deal with the solutions of system (7), seen as mappings $t \mapsto (K(t), L(t))$, for $[0, \infty) \to \Omega$, where

$$\Omega \equiv \{\ (K, L) \in \mathbb{R}^2_{++} : c(K, L) > 0\}.$$

Again, notice that system (7) depends on the parameter vector $\varphi \in \Phi$ (representing endowments, preferences, and technologies), where Φ is model-specific.

In this article, we explore models in which the consumption function takes the specific form

$$C = c(K, L, \varphi) = z\,(K, \mathcal{L}, \varphi)\,,\ \text{with}\ \mathcal{L}\ \equiv L^{\epsilon(\varphi)}\ .$$

Therefore, $c_K(K, L, \varphi) = z_K(K, L^{\epsilon(\varphi)}, \varphi)$ and $c_L(K, L, \varphi) = \epsilon(\varphi) z_{\mathcal{L}}(K, L^{\epsilon(\varphi)}, \varphi) L^{\epsilon(\varphi)-1}$. Using this specification it is convenient to rewrite the system (7) as

$$\begin{aligned} \dot{K} &= f_1(K, L, \varphi) \ , \\ \epsilon(\varphi)\, \delta(K, L, \varphi)\, \dot{L} &= f_2(K, L, \varphi) \ , \end{aligned} \tag{8}$$

where

$$f_1(K, L, \varphi) \equiv s(K, L, \varphi) \equiv y(K, L, \varphi) - z(K, L^{\epsilon(\varphi)}, \varphi) \ , \tag{9}$$

$$f_2(K, L, \varphi) \equiv z(K, L^{\epsilon(\varphi)}, \varphi)\ (R(K, L, \varphi) - \rho)\ \theta\,(K, L, \varphi)\ L^{\epsilon(\varphi)-1}, \tag{10}$$

$$\delta(K, L, \varphi) \equiv z_{\mathcal{L}}(K, L^{\epsilon(\varphi)}, \varphi). \tag{11}$$

Steady-states of system (7) are points $(\bar{K}, \bar{L})$ belonging to set:

$$\Gamma_E \equiv \{\ (K, L) \in \Omega : f_1(K, L, \varphi) = f_2(K, L, \varphi) = 0\}\ ,$$

which can have one element or more. The stationarity conditions are similar to those in the benchmark competitive DGE model

$$s(\bar{K}, \bar{L}, \varphi) = 0, \; r(\bar{K}, \bar{L}, \varphi) = \rho \; ,$$

i.e. steady-state savings are zero and the return on capital equals the rate of time preference, since there is no depreciation.

Let us denote a solution to system (8) for a given time t starting from a point $(K^*, L^*) \in \Omega$ by $\varphi_t(K^*, L^*)$. Define the stable manifold associated to a steady-state $(\bar{K}, \bar{L}) \in \Gamma_E$ as

$$\mathcal{W}^s(\bar{K}, \bar{L}) \equiv \{ \; (K^*, L^*) \in \Omega : \lim_{t\to\infty} \varphi_t(K^*, L^*) = (\bar{K}, \bar{L}) \} \; .$$

Therefore, as a consequence of assumption 1, a DGE path is a trajectory $(K(t), L(t))_{t\in[0,\infty)}$ such that $(K(t), L(t) \in \mathcal{W}^s(\bar{K}, \bar{L})$ for every $t \in [0, \infty)$. In the case in which $\mathcal{W}^s(\bar{K}, \bar{L})$ is empty, the DGE path is stationary and it exists only if $K(0) = \bar{K}$ [3].

We introduce the three following assumptions. First, function $z(\cdot)$ is monotonic in K, which implies that $c_K(K, L, \epsilon) \neq 0$ for all $(K, L, \varphi) \in \Omega \times \Phi$. Second, ϵ is a primitive parameter or a function of the primitive parameters, $\epsilon = \epsilon(\varphi)$, such that it is a small number centered around zero, e.g. $-1 < \epsilon < 1$ or $-1 < \epsilon(\varphi) < 1$. Third, function $z(\cdot)$ is non-monotonic in L and there is one value $L_s = L_s(K, \varphi)$ such that $z_{\mathcal{L}}(K, L_s^\epsilon, \varphi) = 0$.

The two last assumptions imply that the consumption function can have a zero derivative with respect to labor from two different sources: (i) when the parameter vector is such that $\epsilon(\varphi_0) = 0$ or (ii) when $L = L_s(K, \varphi)$. Those are the two main origins of singularities in our model.

Let us introduce the following definitions to distinguish two different types of singularities:

Definition 5. *A point $(K, L, \varphi) \in \Omega \times \Phi$ is a regular point if $\epsilon(\varphi)\,\delta(K, L, \varphi) \neq 0$. A singularity exists if point $(K, L, \varphi) \in \Omega \times \Phi$ is such that $\epsilon(\varphi)\,\delta(K, L, \varphi) = 0$. We call singular perturbation to a point $\varphi^p \in \Phi$ such that $\epsilon(\varphi^p) = 0$. We call impasse set to a subset of points $(K^s, L^s) \in \Omega$ such that $\delta(K^s, L^s, \varphi) = 0$.*

Henceforth, we assume for simplicity that there is either a singular perturbation or impasses, not both:

Assumption 2. *There are no points (K^*, L^*, φ^*) such that $\epsilon(\varphi^*) = 0$ and $\delta(K^*, L^*, \varphi^*) = 0$ simultaneously.*

[3]Observe that we are now referring to a two-dimensional projection in (K, L) of a three-dimensional system in (K, L, C)

Before we characterize DGE paths in the presence of singularities, we can show that they have a fundamentally different nature depending on whether they arise from a singular perturbation or from an impasse.

Proposition 1. *Singular-perturbed steady-states are generic points of set Ω while impasse steady-states are non-generic points of set Ω.*

This is easy to see if we take into account that singular-perturbed steady states belong to set:

$$\Gamma_E^p \equiv \{ \ (K,L) \in \Omega : f_1(K,L,\varphi^p) = f_2(K,L,\varphi^p) = 0\} \ ,$$

where $\epsilon(\varphi^p) = 0$ and impasse steady-states belong to set

$$P_E \equiv \{ \ (K,L,\varphi) \in \Omega \times \Phi : f_1(K,L,\varphi) = f_2(K,L,\varphi) = \delta(K,L,\varphi) = 0\}, \tag{12}$$

in which one parameter should take a particular value.

Proposition 2. *Singular perturbations have a permanent character (i.e. they characterize the entire DGE paths), because they occur when a parameter exhibits a particular critical value (or the parameters satisfy a specific relationship). An impasse singularity generically constrains the existence of DGE paths, or changes their dynamic properties, after a finite time.*

In order to prove this we denote the vector field in system (7) by

$$F(K,L,\varphi) = \begin{pmatrix} f_1(K,L,\varphi) \\ \\ \dfrac{f_2(K,L,\varphi)}{\epsilon(\varphi)\delta(K,L,\varphi)} \end{pmatrix}. \tag{13}$$

The Jacobian of $F(\cdot)$, evaluated at any point $(K,L) \in \Omega$, is

$$DF(K,L,\varphi) = \begin{pmatrix} f_{1,K}(K,L,\varphi) & f_{1,L}(K,L,\varphi) \\ \dfrac{f_{2,K}(K,L,\varphi)}{\epsilon(\varphi)\,\delta(K,L,\varphi)} - \dfrac{f_2(K,L,\varphi)\,\delta_K(K,L,\varphi)}{\epsilon(\varphi)\,\delta^2(K,L,\varphi)} & \dfrac{f_{2,L}(K,L,\varphi)}{\epsilon(\varphi)\,\delta(K,L,\varphi)} - \dfrac{f_2(K,L,\varphi)\,\delta_L(K,L,\varphi)}{\epsilon(\varphi)\,\delta^2(K,L,\varphi)} \end{pmatrix} , \tag{14}$$

has trace and determinant given by

$$\mathrm{tr}DF(K,L,\varphi) = f_{1,K}(K,L,\varphi) + \frac{f_{2,L}(K,L,\varphi)\,\delta(K,L,\varphi) - f_2(K,L,\varphi)\,\delta_L(K,L,\varphi)}{\epsilon(\varphi)\,\delta^2(K,L,\varphi)} \ ,$$

and

$$\det DF(K,L,\varphi) = \frac{f_{1,K}(K,L,\varphi)f_{2,L}(K,L,\varphi) - f_{1,L}(K,L,\varphi)f_{2,K}(K,L,\varphi)}{\epsilon(\varphi)\,\delta(K,L,\varphi)} + \\ + f_2(K,L,\varphi)\left(\frac{f_{1,K}(K,L,\varphi)\delta_L(K,L,\varphi) - f_{1,L}(K,L,\varphi)\delta_K(K,L,\varphi)}{\epsilon(\varphi)\,\delta^2(K,L,\varphi)}\right) \ . \quad (15)$$

There is a point in common between the two types of singularities: generically after the singularity is crossed the dimension of the stable manifold changes because one eigenvalue of $DF(K,L,\varphi)$ changes sign by passing through plus or minus infinity. However, while for singular perturbations the singularity crossing is associated to a change of a parameter away from a critical value, for impasse singularities it is associated to a change in the value of variables (K,L) away from a specific relationship, as we see next.

For singular perturbations we have:

$$\lim_{\varphi\to\varphi^p} \mathrm{tr}DF(K,L,\varphi) = \lim_{\varphi\to\varphi^p} \det DF(K,L,\varphi) = \pm\infty \ ,$$

and if φ_1 and φ_2 belong to the vicinity of φ^p and $\epsilon(\varphi_1) < 0 < \epsilon(\varphi_2)$, then

$$\mathrm{sign}\left(\det DF(K,L,\varphi_1)\right) \neq \mathrm{sign}\left(\det DF(K,L,\varphi_2)\right).$$

For impasse points we have:

$$\lim_{(K,L)\to(K^s,L^s)} \mathrm{tr}DF(K,L,\varphi) = \lim_{(K,L)\to(K^s,L^s)} \det DF(K,L,\varphi) = \pm\infty \ ,$$

and if (K_1,L_1) and (K_2,L_2) belong to a small neighborhood of (K^s,L^s) and $\delta(K_1,L_1,.) < 0 < \delta(K_2,L_2,.)$, then sign($\det DF(K_1,L_1,\varphi)$) $\neq$sign($\det DF(K_2,L_2,\varphi)$).

The Jacobian of $F(\cdot)$ evaluated at a steady-state $(\bar{K},\bar{L}) \in \Gamma_E$, for a given parameter value φ, is

$$DF(\bar{K},\bar{L},\varphi) = \begin{pmatrix} f_{1,K}(\bar{K},\bar{L},\varphi) & f_{1,L}(\bar{K},\bar{L},\varphi) \\ \dfrac{f_{2,K}(\bar{K},\bar{L},\varphi)}{\epsilon(\varphi)\,\delta(\bar{K},\bar{L},\varphi)} & \dfrac{f_{2,L}(K,L,\varphi)}{\epsilon(\varphi)\,\delta(\bar{K},\bar{L},\varphi)} \end{pmatrix} \ , \quad (16)$$

having the trace and the determinant given by

$$\mathrm{tr}DF(\bar{K},\bar{L},\varphi) = f_{1,K}(\bar{K},\bar{L},\varphi) + \frac{f_{2,L}(K,L,\varphi)}{\epsilon(\varphi)\,\delta(K,L,\varphi)} \ ,$$

$$\det DF(\bar{K},\bar{L},\varphi) = \frac{f_{1,K}(\bar{K},\bar{L},\varphi)f_{2,L}(\bar{K},\bar{L},\varphi) - f_{1,L}(\bar{K},\bar{L},\varphi)f_{2,K}(\bar{K},\bar{L},\varphi)}{\epsilon(\varphi)\,\delta(\bar{K},\bar{L},\varphi)} \ . \quad (17)$$

For singular perturbations we still have:

$$\lim_{\varphi \to \varphi^p} \mathrm{tr} DF(\bar{K}, \bar{L}, \varphi) = \lim_{\varphi \to \varphi^p} \det DF(\bar{K}, \bar{L}, \varphi) = \pm\infty.$$

When impasse singularities exist and if the steady-state is not an impasse-singular steady-state, that is if $(\bar{K}, \bar{L}) \notin P_E$ and $\varphi \neq \varphi^p$, then both $\mathrm{tr} DF(\bar{K}, \bar{L}, \varphi)$ and $\det DF(\bar{K}, \bar{L}, \varphi)$ are finite, which means that the eigenvalues are finite and the steady-state is regular.

Therefore, there is a major difference between the two types of singularity. The singular perturbation changes the signs of the eigenvalues of the Jacobian $DF(K, L, \varphi)$ at each and every point $(K, L) \in \Omega$, including at steady-states, when the parameter φ passes through the critical point φ^p. The impasse singularity, in contrast, generates infinitely-valued eigenvalues only when the variables (K, L) satisfy the condition $\delta(K, L) = 0$, which generically occurs at points which are not steady-states, and steady-states are regular points. To put it simply, while DGE paths are everywhere pointwise-singular for singular perturbations, DGE paths are almost everywhere pointwise-regular and are pointwise-singular for a small number of impasse points for impasse singularities.

Next, we enumerate and characterize the types of DGE paths that can occur in the presence of a singularity.

2.3 DGE paths in the presence of singular perturbations

In this section, we assume there is a critical value for φ, $\varphi^p \in \Phi$, such that $\epsilon(\varphi^p) = 0$. From assumption 2 we also have $\delta(K, L, \varphi^p) \neq 0$ for all $(K, L) \in \Omega$. We study the DGE dynamics associated to varying φ in the vicinity of φ^p such that $\epsilon(\varphi) \in (-1, 1)$ and $\delta(K, L, \epsilon(\varphi)) \neq 0$.

For convenience, we write system (8) in the following equivalent form:

$$\begin{aligned} \dot{K} &= f_1(K, L, \epsilon), \\ \epsilon \, \dot{L} &= f_2^s(K, L, \epsilon), \end{aligned} \tag{18}$$

where $f_2^s(K, L, \epsilon) \equiv f_2(K, L, \epsilon)/\delta(K, L, \epsilon)$.

Systems of type (18), with $\epsilon \in (-1, 1)$, are called *singular-perturbed* or *slow-fast* systems[4]. That designation is justified because the two variables have two different time scales: K has a slow time scale, and L has a fast time scale. Time t is called slow time and time $\tau = t/\epsilon$ is called fast time. If ϵ is close to zero we see that the adjustment of L is very fast along a curve $f_2^s(K, L, \epsilon) \approx 0$.

[4]See Kuehn (2015) for a recent textbook presentation.

As time derivatives in system (18) refer to the dynamics along the slow time scale we call it the slow system. The dynamics along the fast time scale is

$$\begin{aligned} K^{'} &= \frac{dK}{d\tau} = \epsilon\, f_1(K, L, \epsilon), \\ L^{'} &= \frac{dL}{d\tau} = f_2^s(K, L, \epsilon). \end{aligned} \tag{19}$$

This pair of systems (18)-(19) is called a *slow-fast system.*

The slow-fast vector fields associated to the slow-fact system (18)- (19) are, respectively,

$$F(K, L, \epsilon) = \begin{pmatrix} f_1(K, L, \epsilon) \\ \dfrac{f_2^s(K, L, \epsilon)}{\epsilon} \end{pmatrix} \text{ and } F^f(K, L, \epsilon) = \begin{pmatrix} \epsilon f_1(K, L, \epsilon) \\ f_2^s(K, L, \epsilon) \end{pmatrix} .$$

If $\epsilon \neq 0$, the dynamics and bifurcations of system (18) are both regular and well-known. If functions $f_1(K, L)$ and $f_2(K, L)$ are differentiable then, given any initial value $K_0 \in \Omega$ a solution to system (18)-(19) exists, it is unique, and it is continuous in t, for all $t \in [0, \infty)$.

Furthermore, at a regular equilibrium point $(\bar{K}, \bar{L}) \in \Gamma_E$ (for $\Gamma_E \neq \varnothing$), the local dynamics is characterized by the eigenvalues of the Jacobian for the slow-system evaluated at that steady-state, $DF(\bar{K}, \bar{L})$. Clearly, there may exist values for ϵ such that one or more eigenvalues can have zero real parts (and therefore regular bifurcation can exist) or can change from real to complex (or back) [5].

The Jacobian of the slow-vector field, F, evaluated at a steady-state

$$DF(\bar{K}, \bar{L}, \epsilon) = \begin{pmatrix} f_{1,K}(\bar{K}, \bar{L}, \epsilon) & f_{1,L}(\bar{K}, \bar{L}, \epsilon) \\ \dfrac{f_{2,K}^s(\bar{K}, \bar{L}, \epsilon)}{\epsilon} & \dfrac{f_{2,L}^s(\bar{K}, \bar{L}, \epsilon)}{\epsilon} \end{pmatrix} ,$$

has trace and determinant given by

$$\text{tr} DF(\bar{K}, \bar{L}, \epsilon) = f_{1,K}(\bar{K}, \bar{L}, \epsilon) + \frac{f_{2,L}^s(\bar{K}, \bar{L}, \epsilon)}{\epsilon} , \tag{20}$$

$$\det DF(\bar{K}, \bar{L}, \epsilon) = \frac{f_{1,K}(\bar{K}, \bar{L}, \epsilon)\, f_{2,L}^s(\bar{K}, \bar{L}, \epsilon) - f_{1,L}(\bar{K}, \bar{L}, \epsilon)\, f_{2,K}^s(\bar{K}, \bar{L}, \epsilon)}{\epsilon} . \tag{21}$$

The local stable manifold is one-dimensional for $\det DF(\bar{K}, \bar{L}, \epsilon) < 0$ and it is two-dimensional (or zero-dimensional) for $\det DF(\bar{K}, \bar{L}, \epsilon) > 0$ and $\text{tr} DF(\bar{K}, \bar{L}, \epsilon) < 0$ (or > 0).

[5] Major references for continuous-time regular dynamics and bifurcations are Guckenheimer and Holmes (1990) or Kuznetsov (2005).

If $\epsilon = 0$ we already know that the eigenvalues of the Jacobian for the slow system evaluated at any point of Ω become infinite. In particular, the eigenvalues of the Jacobian of the slow system evaluated at the steady-state also become infinite because

$$\lim_{\epsilon \to 0} \mathrm{tr} DF(\bar{K}, \bar{L}, \epsilon) = \lim_{\epsilon \to 0} \det DF(\bar{K}, \bar{L}, \epsilon) = \pm\infty.$$

However, the local dynamics in the neighborhood of a singular-perturbation point can be revealed by characterising the dynamics of the fast system in the neighbourhood of a steady-state. For any value of ϵ, the Jacobian of the fast vector field F^s evaluated at a steady-state $(\bar{K}, \bar{L}, \epsilon)$ is

$$DF^s(\bar{K}, \bar{L}, \epsilon) = \begin{pmatrix} \epsilon\, f_{1,K}(\bar{K}, \bar{L}, \epsilon) & \epsilon\, f_{1,L}(\bar{K}, \bar{L}, \epsilon) \\ f^s_{2,K}(\bar{K}, \bar{L}, \epsilon) & f^s_{2,L}(\bar{K}, \bar{L}, \epsilon) \end{pmatrix}.$$

As the trace and determinant are

$$\mathrm{tr} DF^s(\bar{K}, \bar{L}, \epsilon) = \epsilon\, f_{1,K}(\bar{K}, \bar{L}, \epsilon) + f^s_{2,L}(\bar{K}, \bar{L}, \epsilon)\ , \tag{22}$$

$$\det DF^s(\bar{K}, \bar{L}, \epsilon) = \epsilon \left[f_{1,K}(\bar{K}, \bar{L}, \epsilon)\, f^s_{2,L}(\bar{K}, \bar{L}, \epsilon) - f_{1,L}(\bar{K}, \bar{L}, \epsilon)\, f^s_{2,K}(\bar{K}, \bar{L}, \epsilon) \right], \tag{23}$$

then

$$\mathrm{tr} DF^s(\bar{K}, \bar{L}, 0) = f^s_2(\bar{K}, \bar{L}, 0),\ \det DF^s(\bar{K}, \bar{L}, 0) = 0.$$

we see that the fast vector field evaluated in the vicinity of a singular perturbation, $\epsilon = 0$, has the characteristics of a regular-bifurcation point, i.e. the determinant changes sign by passing through zero, instead of passing through infinite as is the case for the slow vector field.

In order to understand the local dynamics when $\epsilon = 0$ it is convenient to write the slow-fast system as

$$\begin{aligned} \dot{K} &= f_1(K, L, 0)\ , & K' &= 0\ . \\ 0 &= f^s_2(K, L, 0)\ , & L' &= f^s_2(K, L, 0)\ . \end{aligned} \tag{24}$$

We define a *singular-perturbed critical subset* by[6]

$$\mathcal{S}^p = \{(K, L) \in \Omega : f_2(K, L, 0) = 0\}\ .$$

We say that point $(K^p, L^p) \in \mathcal{S}^p$ is a *slow-fast singular* for $f^s_{2,L}(K^p, L^p, 0) = 0$. Point $(K^*, L^*) \neq (K^p, L^p) \in \mathcal{S}^p$ is *slow-fast regular* for $f^s_{2,L}(K^*, L^*, 0) \neq 0$.

[6] Recall we are assuming that $\delta(K, L, \epsilon) \neq 0$.

At a slow-fast regular point we can solve equation $f_2^s(K, L, 0) = 0$ for L as a function of K, $L = h(K)$, by applying the implicit-function theorem. Locally the slope of function $h(K)$ is

$$h_K(K) = -\frac{f_{2,K}^s(K, L, 0)}{f_{2,L}^s(K, L, 0)} = -\frac{f_{2,K}(K, L, 0)}{f_{2,L}(K, L, 0)} \text{ for } (K, L) \in \mathcal{S}^p .$$

This means that the dynamics evolves along the surface $\mathcal{S}^p$, which is geometrically represented by the isocline associated to L.

We call slow-fast regular point to a point (K, L) such that $f_{2.L}(K, L, 0) \neq 0$ and slow-fast singular point to a point (K', L') such that $f_{2.L}(K', L', 0) = 0$. If slow-fast singular points do not exist, then that surface contains only one of two types of points: (i) *slow-fast regular attracting* points if $f_{2,L}^s(K, L, 0) = f_{2,L}(K, L, 0)/\delta(K, L, 0) < 0$ or (ii) *slow-fast regular repelling* points if $f_{2,L}^s(K, L, 0) = f_{2,K}(K, L, 0)/\delta(K, L, 0) > 0$, for $(K, L) \in \mathcal{S}^p$.

One of the most important results on the mathematics of slow-fast systems, the Fenichel (1979) theorem, states that in the neighborhood of surface $\mathcal{S}^p$ the dimension of the stable manifold is not changed by a small variation of the parameter ϵ in one of the neighborhoods of $\epsilon = 0$, that is either when $\epsilon \to 0^+$ or when $\epsilon \to 0^-$.

We denote the steady-state of system (24) by $(\bar{K}^p, \bar{L}^p)$. A taxonomy for the generic local dynamics in a neighborhood of a steady-state for system (18)-(19) can be built:

1. Let $f_{1,K}(\bar{K}, \bar{L}, \epsilon) f_{2,L}^s(\bar{K}, \bar{L}, \epsilon) - f_{1,L}(\bar{K}, \bar{L}, \epsilon) f_{2,K}^s(\bar{K}, \bar{L}, \epsilon) > 0$ in a small neighborhood of ϵ centred around zero, i.e., for $\epsilon \in (0^-, 0^+)$:

    (a) If $f_{2,L}^s(\bar{K}, \bar{L}, \epsilon) < 0$, then (i) the steady state is a stable node for $\epsilon \to 0^+$, as we have $\text{tr}DF^s(\bar{K}, \bar{L}, 0^+) < 0$ and $\det DF^s(\bar{K}, \bar{L}, 0^+) > 0$; (ii) set $\mathcal{S}^p$ only contains slow-fast regular attractor points converging to steady-state $(\bar{K}^p, \bar{L}^p)$ for $\epsilon = 0$, since $f_{2,L}^s(\bar{K}^p, \bar{L}^p, 0) < 0$; and (iii) the regular steady-state is a saddle point for $\epsilon \to 0^-$, as $\det DF^s(\bar{K}, \bar{L}, 0^-) < 0$.

    (b) If $f_{2,L}^s(\bar{K}, \bar{L}, \epsilon) > 0$, then (i) the steady state is an unstable node for $\epsilon \to 0^+$, as we have $\text{tr}DF^s(\bar{K}, \bar{L}, 0^+) > 0$ and $\det DF^s(\bar{K}, \bar{L}, 0^+) > 0$; (ii) set $\mathcal{S}^p$ only contains slow-fast regular repeller points diverging from steady-state $(\bar{K}^p, \bar{L}^p)$ for $\epsilon = 0$, since $f_{2,L}^s(\bar{K}^p, \bar{L}^p, 0) > 0$; and (iii) the regular steady-state is a saddle point for $\epsilon \to 0^-$, as $\det DF^s(\bar{K}, \bar{L}, 0^-) < 0$.

2. For $f_{1,K}(\bar{K}, \bar{L}, \epsilon) f_{2,L}^s(\bar{K}, \bar{L}, \epsilon) - f_{1,L}(\bar{K}, \bar{L}, \epsilon) f_{2,K}^s(\bar{K}, \bar{L}, \epsilon) < 0$ in a small neighborhood of ϵ centred around zero, i.e. for $\epsilon \in (0^-, 0^+)$:

(a) If $f^s_{2,L}(\bar{K},\bar{L},\epsilon)<0$, then (i) the steady state is a saddle point for $\epsilon\to 0^+$, as we have $\det DF^s(\bar{K},\bar{L},0^+)<0$; (ii) set $\mathcal{S}^p$ only contains slow-fast regular attractor points converging to steady-state $(\bar{K}^p,\bar{L}^p)$ for $\epsilon=0$, since $f^s_{2,L}(\bar{K}^p,\bar{L}^p,0)<0$; and (iii) the regular steady-state is an unstable node for $\epsilon\to 0^-$, as $\mathrm{tr}DF^s(\bar{K},\bar{L},0^-)>0$ and $\det DF^s(\bar{K},\bar{L},0^-)>0$.

(b) If $f^s_{2,L}(\bar{K},\bar{L},\epsilon)>0$, then (i) the steady state is a saddle point for $\epsilon\to 0^+$, as $\det DF^s(\bar{K},\bar{L},0^+)<0$; (ii) set $\mathcal{S}^p$ only contains slow-fast regular repeller points diverging from steady-state $(\bar{K}^p,\bar{L}^p)$ for $\epsilon=0$, since $f^s_{2,L}(\bar{K}^p,\bar{L}^p,0)>0$; and (iii) the regular steady-state is a stable node for $\epsilon\to 0^-$, as $\mathrm{tr}DF^s(\bar{K},\bar{L},0^-)<0$ and $\det DF^s(\bar{K},\bar{L},0^-)>0$.

We can summarize the previous discussion in the following Proposition 3 which describes the types of DGE paths that can exist in the presence of a singular perturbation:

Proposition 3 (DGE paths in the presence of a singular perturbation)**.** *Assume that there is a singular-perturbation and that $K_0\in\mathcal{W}^s(\bar{K},\bar{L})$, if $\mathcal{W}^s(\bar{K},\bar{L})$ is non-empty, or that $K_0=\bar{K}$, if $\mathcal{W}^s(\bar{K},\bar{L})$ is empty. Then, only two generic cases exist:*

1. *If the DGE path exhibits permanent indeterminacy for $\epsilon\to 0^+$ ($\epsilon\to 0^-$), it is determinate for both $\epsilon=0$ and $\epsilon\to 0^-$ ($\epsilon=0^+$).*
2. *If the DGE is stationary for $\epsilon\to 0^+$ ($\epsilon\to 0^-$), it is still stationary for $\epsilon=0$ and it is determinate for $\epsilon=0^-$ ($\epsilon\to 0^+$).*

The cases presented above always produce stable or unstable nodes, but not stable or unstable foci. This is due to the fact that in generic cases, for $\epsilon\approx 0$, the trace of the Jacobian $DF(\bar{K},\bar{L},0^{\pm})$ becomes very large in absolute value, which implies that the eigenvalues have to be real. However, for values of ϵ in a wider range around zero, the trace of the Jacobian tends to decrease which implies that the discriminant

$$\Delta DF(\bar{K},\bar{L},\epsilon)=\frac{1}{4}\left[\left(f_{1,K}(\bar{K},\bar{L},\epsilon)-\frac{f^s_{2,L}(\bar{K},\bar{L},\epsilon)}{\epsilon}\right)^2-\frac{4\,f_{1,L}(\bar{K},\bar{L},\epsilon)\,f^s_{2,K}(\bar{K},\bar{L},\epsilon)}{\epsilon}\right]$$

can become negative, and eigenvalues may become complex, leading to oscillatory dynamics.

A geometric argument for the existence of real eigenvalues close to the singular critical set containing only slow-fast regular points is that the solution path will tend to evolve along the isocline for L, where $f_2(K,L,0)=0$, which is a monotonic surface.

Whilst we have only considered the cases in which slow-fast singular points exists for non-zero $f_{2,L}(K, L, 0)$, there are also further results which hold for the zero case (for bifurcation results in slow-fast systems for the zero case, see Kuehn (2015, ch 3, ch 8)).

2.4 DGE paths in the presence of impasse singularities

In this section we assume that $\epsilon(\varphi) \neq 0$ for all $\varphi \in \Phi$ and define the *impasse set* as

$$\mathcal{S} = \{(K, L) \in \Omega : \delta(K, L, \varphi) = 0\} \ . \tag{25}$$

Assumption 2 implies that set $\mathcal{S}$ is non-empty, which means that it introduces a partition over state Ω, such that $\Omega = \Omega_- \cup \mathcal{S} \cup \Omega_+$, where we define:

$$\Omega_- \equiv \{(K, L) : \delta(K, L, \varphi) < 0\} \text{ and } \Omega_+ \equiv \{(K, L) : \delta(K, L, \varphi) > 0\},$$

which are both open subsets containing exclusively regular points.

From now on we assume there are only regular impasse points, that is points $(K, L) \in \mathcal{S}$ such that $\nabla\delta(K, L) \neq 0$. Singular impasse points are points satisfying $\nabla\delta(K, L) = 0$.

Zhitomirskii (1993) and Llibre, Sotomayor, and Zhitomirskii (2002) prove that regular impasse points $(K^s, L^s) \in \mathcal{S}$, such that $\delta(K^s, L^s) = 0$, $\nabla\delta(K^s, L^s) \neq 0$, and $f_1(K^s, L^s) \neq 0$, can be of the following types [7]:

1. $(K^s, L^s) \in \mathcal{S}$ is an *impasse-repeller point* if $\delta_L(K^s, L^s, \varphi) \neq 0$, $f_2(K^s, L^s, \varphi) \neq 0$ and $\delta_L(K^s, L^s, \varphi)\, f_2(K^s, L^s, \varphi) > 0$. At an impasse-repeller point there are two trajectories that are repelled away from $\mathcal{S}$, one to the interior of Ω_- and another to the interior of Ω_+. This means that, if those paths are solutions of system (8) they can, in most generic cases, be continued until $t \to \infty$, thus satisfying a necessary condition for being DGE paths. We denote the sets of impasse-repeller points by $\mathcal{S}_+ \equiv \{(K, L) \in \mathcal{S} :\ \delta_L(K, L, \varphi)\, f_2(K, L, \varphi) > 0\}$.

2. $(K^s, L^s) \in \mathcal{S}$ is an *impasse-attractor point* if $\delta_L(K^s, L^s, \varphi) \neq 0$, $f_2(K^s, L^s, \varphi) \neq 0$ and $\delta_L(K^s, L^s, \varphi)\, f_2(K^s, L^s, \varphi) < 0$. An impasse-attractor point attracts two trajectories, one coming from the interior of Ω_- and another from the interior of Ω_+. This means that, if those paths are solutions of system (8) they are only defined for $t \in [0, t^s)$, where t^s is the time of collision with $\mathcal{S}$. Therefore, they cannot be DGE paths, as

[7] A singular impasse point is a point in $\mathcal{S}$ such that $\nabla\delta = 0$.

these trajectories cannot be continued until $t \to \infty$. We denote the sets of impasse-attractor points by $\mathcal{S}_- \equiv \{(K,L) \in \mathcal{S} : \delta_L(K,L,\varphi)\, f_2(K,L,\varphi) < 0\}$.

3. $(K^s, L^s) \in \mathcal{S}$ is a *impasse-tangent point* if $\delta_L(K^s, L^s, \varphi) = 0$ and $f_2(K^s, L^s, \varphi) \neq 0$. At such a point, there is one trajectory coming from the interior of Ω_+ (or Ω_-) that is tangent to $\mathcal{S}$ at $t = t^s < \infty$ and has a continuation in the interior of the same subset Ω_+ (or Ω_-). This means that, if those paths are solutions of system (8) they can, in most generic cases, be continued until $t \to \infty$ within the same subset Ω_+ (or Ω_-). Thus, a necessary condition for being considered DGE paths holds. We denote the set of impasse-tangent points by $\Gamma_K \equiv \{(K,L) \in \mathcal{S} : \delta_L(K,L,\varphi) = 0,\ f_2(K,L,\varphi) \neq 0\}$.

4. $(K^s, L^s) \in \mathcal{S}$ is a *impasse-transversal point* if $\delta_L(K^s, L^s, \varphi) \neq 0$ and $f_2(K^s, L^s, \varphi) = 0$. We denote the set of these points by $\Gamma_I \equiv \{(K,L) \in \mathcal{S} : f_2(K,L,\varphi) = 0,\ \delta_L(K,L,\varphi) \neq 0\}$. Trajectories that cross through impasse-transversal may or may not exist. In other words, we may obtain trajectories that originate in the interior of one or both subsets, Ω_+ or Ω_-, are transversal to $\mathcal{S}$, and have a continuation for $t \in (t^s, \infty)$ in the interior of the other subset. This means that, we may have two additional cases: (i) paths that remain within the same subset Ω_+ or Ω_- for all $t \in [0, \infty)$ and satisfy a necessary condition to be DGE paths and (ii) paths colliding with $\mathcal{S}$, but which have no continuation cannot be DGE paths, as solutions do not exist for $t \in (t_s, \infty)$. Zhitomirskii (1993) demonstrates that three types of impasse-transversal points exist: (i) *impasse-transversal nodes*, at which an infinite number of trajectories crossing $\mathcal{S}$ exist, both traveling from the interior of one subset (Ω_+ or Ω_-) to the interior of the other subset and no trajectories going in the opposite direction exist; (ii) *impasse-transversal saddles*, at which two trajectories crossing $\mathcal{S}$ exist, one traveling from the interior of Ω_+ to the interior of Ω_- and another traveling from the interior of Ω_- to the interior of Ω_+; and (iii) *impasse-transversal foci*, at which there can be no crossing of $\mathcal{S}$.

In order to determine the type of a impasse-transversal point, it is convenient to consider the dynamics for the desingularized system bellow:

$$\begin{aligned} \dot{K} &= \epsilon(\varphi)\, \delta(K,L,\varphi)\, f_1(K,L,\varphi)\ , \\ \dot{L} &= f_2(K,L,\varphi)\ , \end{aligned} \tag{26}$$

which has the same integral curves as system (8), but in which the direction field within subspace Ω_- is inverted and the singularity at $\mathcal{S}$ is removed. We define the desingularized vector field for system (26) as

$$F^r(K,L,\varphi) = \begin{pmatrix} \epsilon(\varphi)\,\delta(K,L,\varphi)\,f_1(K,L,\varphi) \\ f_2(K,L,\varphi) \end{pmatrix} . \tag{27}$$

By denoting impasse-transversal point as $(K^i, L^i) \in \Gamma_I$, we can readily see that they are fixed points of the desingularized vector field, due to the fact that $\delta(K^i, L^i, \varphi) = f_2(K^i, L^i, \varphi) = 0$, i.e. $F^r(K^i, L^i, \varphi) = \mathbf{0}$, where $\delta_L(K^i, L^i, \varphi) \neq 0$. Therefore, the Jacobian of the desingularized vector field $F^r(\cdot)$, evaluated at (K^i, L^i), is given by

$$DF^r(K^i,L^i,\varphi) = \begin{pmatrix} \epsilon(\varphi)\,\delta_K(K^i,L^i,\varphi)\,f_1(K^i,L^i,\varphi) & \epsilon(\varphi)\,\delta_L(K^i,L^i,\varphi)\,f_1(K^i,L^i,\varphi) \\ f_{2,K}(K^i,L^i,\varphi) & f_{2,L}(K^i,L^i,\varphi) \end{pmatrix} . \tag{28}$$

Its trace and determinant are, respectively,

$$\mathrm{tr}DF^r(K^i,L^i,\varphi) = \epsilon(\varphi)\,\delta_K(K^i,L^i,\varphi)\,f_1(K^i,L^i,\varphi) + f_{2,L}(K^i,L^i,\varphi) \ ,$$

and

$$\begin{aligned} &\det DF^r(K^i,L^i,\varphi) = \\ &\quad = \epsilon(\varphi)\,f_1(K^i,L^i,\varphi)\left(\delta_K(K^i,L^i,\varphi)\,f_{2,L}(K^i,L^i,\varphi) - \delta_L(K^i,L^i,\varphi)\,f_{2,K}(K^i,L^i,\varphi)\right) \ , \end{aligned} \tag{29}$$

which implies that the discriminant is

$$\Delta DF^r(K^i,L^i,\varphi) = (\mathrm{tr}DF^r)^2(K^i,L^i,\varphi) - \det DF^r(K^i,L^i,\varphi) \ .$$

The eigenvalues of Jacobian (28) allows us to determine the type of the impasse-transversal point:

1. (K^i, L^i) is an *impasse-transversal node* if the eigenvalues of (28) are both real and share the same sign, i.e. if (K^i, L^i) belongs to $\Gamma_{IN} \equiv \{(K,L) \in \Gamma_I : \det DF^r(K,L,\varphi) > 0, \Delta DF^r(K,L,\varphi) > 0\}$.
2. (K^i, L^i) is an *impasse-transversal saddle* if the eigenvalues of (28) are both real and exhibit opposite signs, i.e. if (K^i, L^i) belongs to $\Gamma_{IS} \equiv \{(K,L) \in \Gamma_I : \det DF^r(K,L,\varphi) < 0\}$.

3. (K^i, L^i) is an *impasse-transversal focus* if the eigenvalues of (28) are complex-conjugate, i.e. is if (K^i, L^i) belongs to $\Gamma_{IF} \equiv \{(K,L) \in \Gamma_I : \det DF^r(K,L,\varphi) > 0, \Delta DF^r(K,L,\varphi) < 0\}$.

Let us consider that (K^i, L^i) is an impasse-transversal point and $\mathcal{W}^s(K^i, L^i)$ is the stable manifold associated to it. We denote by $\mathcal{W}^s_+(K^i, L^i)$ the stable sub-manifold which is contained in Ω_+ and by $\mathcal{W}^s_-(K^i, L^i)$ the stable sub-manifold which is contained in Ω_-.

If point (K^i, L^i) is an impasse-transversal node and $\mathrm{tr}DF^r(K^i, L^i, .) < 0$ (> 0), then it is attracting (repelling) from side Ω_+ (Ω_-) and it is repelling from side Ω_- (Ω_+). Thus, there is an infinite number of trajectories coming from the interior of Ω_+ (Ω_-) that are attracted to point (K^i, L^i) and are all repelled from (K^i, L^i) to the interior of Ω_- (Ω_+). Furthermore, the basin of attraction of (K^i, L^i) is a two-dimensional stable manifold contained in subset Ω_+ (Ω_-), i.e. $\mathcal{W}^s(K^i, L^i) = \mathcal{W}^s_+(K^i, L^i)$ $(\mathcal{W}^s(K^i, L^i) = \mathcal{W}^s_-(K^i, L^i))$.

If point (K^i, L^i) is a impasse-transversal saddle, there are only two trajectories converging to (K^i, L^i) in finite time, one converging from the interior or Ω_+ and another converging from the interior of Ω_-. These trajectories belong to different integral curves passing through (K^i, L^i), which means that the eigenspaces tangent to the stable manifolds, from both sides Ω_+ and Ω_-, are not collinear. All the remaining integral curves passing trough (K^i, L^i) contain repelling trajectories to the interior of Ω_+ or Ω_-. Therefore, the stable manifold associated with (K^i, L^i) has two one-dimensional branches $\mathcal{W}^s_+(K^i, L^i) \subset \Omega_+$ and $\mathcal{W}^s_-(K^i, L^i) \subset \Omega_-$, i.e. $\mathcal{W}^s(K^i, L^i) = \mathcal{W}^s_+(K^i, L^i) \cup \mathcal{W}^s_-(K^i, L^i)$.

While impasse-repeller and impasse-attractor points belong to a surface in (K, L)[8], the sets of impasse-tangent and -transversal points are isolated points in (K, L)[9]. If no impasse-tangent or -transversal points exist, then set $\mathcal{S}$ only contains repeller or attractor points. To put it simply, if both Γ_I and Γ_K are empty, then only one of two cases is possible: either $\mathcal{S} = \mathcal{S}_+$ and $\mathcal{S}_- = \varnothing$, or $\mathcal{S} = \mathcal{S}_-$ and $\mathcal{S}_+ = \varnothing$. However, if an impasse-tangent or -transversal point exists, then the impasse set contains two open subsets of impasse-repeller and impasse-attractor points. There are two cases: (i) if Γ_I is empty and Γ_K is not, then $\mathcal{S} = \mathcal{S}_- \cup \Gamma_K \cup \mathcal{S}_+$ or (ii) if Γ_K is empty and Γ_I is not, then $\mathcal{S} = \mathcal{S}_- \cup \Gamma_I \cup \mathcal{S}_+$.

The possibility of crossing through the impasse set $\mathcal{S}$ is another important consequence from the existence of impasse-transversal points. This crossing behavior cannot exist in models without impasse singularities.

[8] Indeed, a one-dimensional manifold.

[9] A zero-dimensional manifold.

Lemma 1 (Existence of crossing trajectories)**.** *Trajectories crossing the impasse surface $\mathcal{S}$, from the interior of Ω_+ or Ω_-, can only exist if there is a impasse-transversal point that is either an impasse-transversal node or an impasse-transversal saddle, i.e. if Γ_{IN} or Γ_{IS} are non-empty*[10].

Non-generic regular impasse points are points $(K, L, \varphi) \in (\Gamma_I \cup \Gamma_K) \times \Phi$ where a parameter (or relation between parameters) satisfies a critical condition. These points correspond to one-parameter impasse bifurcations - for a complete characterization see Llibre, Sotomayor, and Zhitomirskii, 2002. One example are points satisfying $\delta(K, L, \varphi) = f_1(K, L, \varphi) = f_2(K, L, \varphi) = 0$, which are members of the set P_E defined in equation (12). Clearly $P_E = \Gamma_E \cap \Gamma_I$. Non-generic points belonging to P_E, satisfying further critical conditions on the parameters, are sometimes called *singularity-induced bifurcation points.*

We call *singular steady-state* to a fixed point located on the impasse surface, $\mathcal{S}$, i.e. to a member of set P_E. Therefore, a *regular steady-state* is a member of the set $\Gamma_E \backslash P_E$. No singular steady-states exist if P_E is empty, although impasse-transversal points (i.e. non-empty Γ_I) may exist.

Next, we assume that there is a regular steady-state $(\bar{K}, \bar{L}) \in \Gamma_E$ in the neighbourhood of an impasse surface $\mathcal{S}$. We will see next that the types of impasse points not only determine which types of steady-states can exist, depending on their location as regards $\mathcal{S}$, but also the confinement, or not, of the stable manifold $\mathcal{W}^s(\bar{K}, \bar{L})$ to one of the subspaces (Ω_+ or Ω_-).

We start with cases in which $\mathcal{S}$ only contains *generic regular impasse points*, such that $\delta_L(K^s, L^s)\, f_2(K^s, L^s) \neq 0$: that is $\mathcal{S}$ contains only impasse-repeller or impasse-attractor points.

Lemma 2 (Local dynamics in the neighborhood of $\mathcal{S}$ containing only generic points)**.** *Assume there is a regular steady-state in a neighborhood of set $\mathcal{S}$ such that $\delta_L(K^s, L^s)\, f_2(K^s, L^s) \neq 0$ for all $(K^s, L^s) \in \mathcal{S}$.*

1. *If the impasse set contains only impasse-attractor points, i.e., if $\mathcal{S} = \mathcal{S}_-$, then the stable manifold $\mathcal{W}^s(\bar{K}, \bar{L})$ is empty or it is one-dimensional;*

2. *If the impasse set contains only impasse-repeller points, i.e., if $\mathcal{S} = \mathcal{S}_+$, then the stable manifold $\mathcal{W}^s(\bar{K}, \bar{L})$ is one- or it is two-dimensional;*

3. *If the stable manifold is not empty it is allways contained in same subset as the steady-state: if $(\bar{K}, \bar{L}) \in \Omega_+$, then $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_+(\bar{K}, \bar{L}) \subset \Omega_+$ and if $(\bar{K}, \bar{L}) \in \Omega_-$, then $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_-(\bar{K}, \bar{L}) \subset \Omega_-$;*

[10]For a proof, see Cardin, Silva, and Teixeira (2012) *inter alia.*

Proof. If a steady-state exists in a neighbourhood of an impasse-attractor point then it is either a saddle point or an unstable node or an unstable focus. If a steady-state exists in a neighbourhood of an impasse-repeller point then it is either a saddle point or an stable node or a stable focus. Because there are no crossing trajectories through $\mathcal{S}$, if it only contains attractor or repeller points, and if the steady-state is a saddle point or is a stable node or focus, then the stable manifold $\mathcal{W}^s(\bar{K}, \bar{L})$ is a subset of the subspace containing the steady-state $(\bar{K}, \bar{L})$. □

Next we assume that the impasse set $\mathcal{S}$ contains only one *non-generic regular impasse point* that partitions it into two open sets of impasse-repeller or impasse-attractor points.

Lemma 3 (Local dynamics in the neighborhood of $\mathcal{S}$ containing one non-generic point)**.** *Assume there is one regular steady-state $(\bar{K}, \bar{L})$ and that set $\mathcal{S}$ contains one non-generic impasse points such that $\delta_L(K^s, L^s)\, f_2(K^s, L^s) = 0$ for $(K^s, L^s) \in \mathcal{S}$. Then:*

1. *If there is one impasse-tangent point $(K^k, L^k) \in \Gamma_K$ then the stable manifold $\mathcal{W}^s(\bar{K}, \bar{L})$ can be empty, one-dimensional or two-dimensional. In the last two cases it is contained in the interior (or in the closure containing the impasse-tangent point) of the subspace containing the steady-state: if $(\bar{K}, \bar{L}) \in \Omega_+$ then $\mathcal{W}^s(\bar{K}, \bar{L}) \supseteq \mathcal{W}^s_+(\bar{K}, \bar{L})$, or if $(\bar{K}, \bar{L}) \in \Omega_-$ then $\mathcal{W}^s(\bar{K}, \bar{L}) \supseteq \mathcal{W}^s_-(\bar{K}, \bar{L})$.*

2. *If there is one impasse-transversal focus point $(K^i, L^i) \in \Gamma_{IF}$ then the stable manifold $\mathcal{W}^s(\bar{K}, \bar{L})$ is one-dimensional and it belongs to the subspace containing the steady-state: if $(\bar{K}, \bar{L}) \in \Omega_+$ then $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_+(\bar{K}, \bar{L})$, or if $(\bar{K}, \bar{L}) \in \Omega_-$ then $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_-(\bar{K}, \bar{L})$.*

3. *If there is one impasse-transversal saddle point $(K^i, L^i) \in \Gamma_{IS}$ either the stable manifold $\mathcal{W}^s(\bar{K}, \bar{L})$ is empty or it is non-empty. If it is non-empty two cases are possible: (i) it can be contained in the interior of one subspace, that is if $(\bar{K}, \bar{L}) \in \Omega_+$, then $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_+(\bar{K}, \bar{L})$, or if $(\bar{K}, \bar{L}) \in \Omega_-$, then $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_-(\bar{K}, \bar{L})$ and it is two-dimensional; (ii) it can have elements in the two subspaces, that is $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_+(\bar{K}, \bar{L}) \cup \{(K^i, L^i)\} \cup \mathcal{W}^s_-(\bar{K}, \bar{L})$ such that the sub-manifold coinciding with the stable manifold associated to (K^i, L^i) is one-dimensional and the sub-manifold belonging to the same space of $(\bar{K}, \bar{L})$ is two-dimensional.*

4. *If there is one impasse-transversal node $(K^i, L^i) \in \Gamma_{IN}$ the stable manifold $\mathcal{W}^s(\bar{K}, \bar{L})$ is allways non-empty. Two cases are possible: (i) it can be contained in the interior of one subspace, that is if $(\bar{K}, \bar{L}) \in \Omega_+$, then $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_+(\bar{K}, \bar{L})$, or if $(\bar{K}, \bar{L}) \in \Omega_-$, then $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_-(\bar{K}, \bar{L})$ and it is one-dimensional; (ii) it can have elements in the two subspaces, that is $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_+(\bar{K}, \bar{L}) \cup \{(K^i, L^i)\} \cup \mathcal{W}^s_-(\bar{K}, \bar{L})$ such that the sub-manifold coinciding with the stable manifold associated to (K^i, L^i) is two-dimensional and the sub-manifold belonging to the same space of $(\bar{K}, \bar{L})$ is one-dimensional.*

Proof. Assume $\mathcal{S}$ contains one impasse-tangent point, i.e. $(K^k, L^k) \in \Gamma_K$. We proved that $\mathcal{S} = \mathcal{S}_+ \cup \Gamma_K \cup \mathcal{S}_-$ and that there is one trajectory which is tangent to $\mathcal{S}$ in finite time. Since the steady-state is regular, we have $(\bar{K}, \bar{L}) \neq (K^k, L^k)$. However, an integral curve passing through $(\bar{K}, \bar{L})$ and (K^k, L^k) may exist, along which trajectories may converge to or diverge from $(\bar{K}, \bar{L})$. In addition, from Lemma 2, we can distinguish two cases: (i) if the steady-state $(\bar{K}, \bar{L})$ is close to $\mathcal{S}_-$, it is an unstable node, an unstable focus or a saddle point or (ii) if it is close to $\mathcal{S}_+$, it is a saddle point, a stable node or a stable focus. In the first case, the tangent trajectory diverges from the steady-state and in the second case, it can be converging or diverging. In any case, no crossing of surface $\mathcal{S}$ is possible. Therefore, assuming that the stable manifold is non-empty, if the tangent trajectory is diverging, then either $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_+(\bar{K}, \bar{L})$ or $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_-(\bar{K}, \bar{L})$, and if the trajectory is converging, then either $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_+(\bar{K}, \bar{L}) \cup \{(K^k, L^k)\}$ or $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_-(\bar{K}, \bar{L}) \cup \{(K^k, L^k)\}$.

Assume $\mathcal{S}$ contains one impasse-transversal focus point, i.e. $(K^i, L^i) \in \Gamma_{IF}$. We proved that $\mathcal{S} = \mathcal{S}_+ \cup \Gamma_{IF} \cup \mathcal{S}_-$ and that there are no trajectories crossing $\mathcal{S}$ in finite time. Since the steady-state is regular, we have $(\bar{K}, \bar{L}) \neq (K^i, L^i)$. Applying Lemma 2 we potentially have the following cases: (i) if the steady-state $(\bar{K}, \bar{L})$ is close to $\mathcal{S}_-$, it can be an unstable node or focus or a saddle point or (ii) if it is close to $\mathcal{S}_+$, it can be a saddle point or a stable node or focus. However, from Lemma 4 the steady-state can only be a saddle point. Considering that there are no trajectories crossing the surface $\mathcal{S}$ then, if the stable manifold is non-empty, either $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_+(\bar{K}, \bar{L})$ or $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_-(\bar{K}, \bar{L})$ and $\mathcal{W}^s(\bar{K}, \bar{L})$ it is one-dimensional.

Assume $\mathcal{S}$ contains one impasse-transversal saddle point, i.e. $(K^i, L^i) \in \Gamma_{IS}$. We proved that $\mathcal{S} = \mathcal{S}_+ \cup \Gamma_{IS} \cup \mathcal{S}_-$ and that (see Lemma 1) there are two non-collinear converging trajectories, originated in both subsets Ω_+ and Ω_-, and passing through (K^i, L^i). Associated to the trajectories passing through (K^i, L^i) there is a one-dimensional stable manifold, spanning the two subspaces Ω_+ and Ω_- , and therefore $\mathcal{W}^s(K^i, L^i) = \mathcal{W}^s_+(K^i, L^i) \cup \mathcal{W}^s_-(K^i, L^i)$. Since the steady-state is regular, we have $(\bar{K}, \bar{L}) \neq (K^i, L^i)$. However, an integral curve passing through $(\bar{K}, \bar{L})$ and (K^i, L^i) can exist, along which trajectories may converge to or diverge from $(\bar{K}, \bar{L})$. Again, using Lemmas 2 and 4 we have: (i) if the steady-state $(\bar{K}, \bar{L})$ is close to $\mathcal{S}_-$, it is an unstable node or an unstable focus, or (ii) if it is close to $\mathcal{S}_+$ it is a stable node or a stable focus. An important distinction is related to the direction of trajectories flowing along integral curves passing through the two points $(\bar{K}, \bar{L})$ and (K^i, L^i): trajectories may diverge from $(\bar{K}, \bar{L})$ or converge to it. In the first case, the stable manifolds $\mathcal{W}^s_+(K^i, L^i)$ and $\mathcal{W}^s(\bar{K}, \bar{L})$ are disjoint and in the second case, they exhibit a non-empty intersection. This allows us enumerate the characteristics of the stable manifold $\mathcal{W}^s(\bar{K}, \bar{L})$, when it is non-empty. If integral curves passing through (K^i, L^i) and $(\bar{K}, \bar{L})$ diverge from $(\bar{K}, \bar{L})$ and the stable manifold is non-empty, then either $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_+(\bar{K}, \bar{L})$ or $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_-(\bar{K}, \bar{L})$ and $\mathcal{W}^s(\bar{K}, \bar{L})$ is two-dimensional. If integral curves passing through (K^i, L^i) and $(\bar{K}, \bar{L})$ converge to $(\bar{K}, \bar{L})$ and the stable manifold is non-empty, then the stable manifold $\mathcal{W}^s(\bar{K}, \bar{L})$ contains points in both subsets Ω_+ and Ω_-, i.e. $\mathcal{W}^s(\bar{K}, \bar{L}) = \mathcal{W}^s_-(\bar{K}, \bar{L}) \cup \{(K^i, L^i)\} \cup \mathcal{W}^s_+(\bar{K}, \bar{L})$.

Thus, two cases are possible: (i) if $(\bar{K},\bar{L}) \in \Omega_+$, then $\mathcal{W}^s_-(\bar{K},\bar{L}) = \mathcal{W}^s_-(K^i,L^i)$ is one-dimensional and $\mathcal{W}^s_+(\bar{K},\bar{L}) \not\subset \mathcal{W}^s_+(K^i,L^i)$ is two-dimensional and (ii) if $(\bar{K},\bar{L}) \in \Omega_-$, then $\mathcal{W}^s_+(\bar{K},\bar{L}) = \mathcal{W}^s_+(K^i,L^i)$ is one-dimensional and $\mathcal{W}^s_-(\bar{K},\bar{L}) \not\subset \mathcal{W}^s_-(K^i,L^i)$ is two-dimensional.

Assume $\mathcal{S}$ contains one impasse-transversal node point, i.e. $(K^i,L^i) \in \Gamma_{IN}$. we proved that $\mathcal{S} = \mathcal{S}_+ \cup \Gamma_{IN} \cup \mathcal{S}_-$ and that (see Lemma 1) there is an infinite number of trajectories coming from only one subspaces, Ω_+ or Ω_-, and passing through (K^i,L^i). There is a two-dimensional stable manifold associated to (K^i,L^i), $\mathcal{W}^s(K^i,L^i) = \mathcal{W}^s_+(K^i,L^i) \subseteq \Omega_+$ or $\mathcal{W}^s(K^i,L^i) = \mathcal{W}^s_-(K^i,L^i) \subseteq \Omega_-$. Since the steady-state is regular, we have $(\bar{K},\bar{L}) \neq (K^i,L^i)$, but there is an integral curve passing through $(\bar{K},\bar{L})$ and (K^i,L^i), over which trajectories may converge to or diverge from $(\bar{K},\bar{L})$. Lemmas 2 and 4 imply that the steady-state will always be a saddle point independently from being close to $\mathcal{S}_-$ or to $\mathcal{S}_+$. Two main cases can be distinguished: First, if both $(\bar{K},\bar{L})$ and $\mathcal{W}^s(K^i,L^i)$ are in the same subset, Ω_+ or Ω_-, or if they are in two different subsets, but along any integral curve joining (K^i,L^i) and $(\bar{K},\bar{L})$, trajectories do not converge to $(\bar{K},\bar{L})$, then either $\mathcal{W}^s(\bar{K},\bar{L}) = \mathcal{W}^s_+(\bar{K},\bar{L})$ or $\mathcal{W}^s(\bar{K},\bar{L}) = \mathcal{W}^s_-(\bar{K},\bar{L})$ and $\mathcal{W}^s(\bar{K},\bar{L})$ is one-dimensional. Second, if $(\bar{K},\bar{L})$ and $\mathcal{W}^s(K^i,L^i)$ are in different subsets and there are integral curves joining (K^i,L^i) and $(\bar{K},\bar{L})$ where trajectories converge to $(\bar{K},\bar{L})$, then the stable manifold $\mathcal{W}^s(\bar{K},\bar{L})$ contains points in both subsets, Ω_+ and Ω_-, i.e. $\mathcal{W}^s(\bar{K},\bar{L}) = \mathcal{W}^s_-(\bar{K},\bar{L}) \cup \{(K^i,L^i)\} \cup \mathcal{W}^s_+(\bar{K},\bar{L})$. Two cases are possible: (i) if $(\bar{K},\bar{L}) \in \Omega_+$, then $\mathcal{W}^s_-(\bar{K},\bar{L}) = \mathcal{W}^s_-(K^i,L^i)$ is two-dimensional and $\mathcal{W}^s_+(\bar{K},\bar{L})$ is one-dimensional, or (ii) if $(\bar{K},\bar{L}) \in \Omega_-$, then $\mathcal{W}^s_+(\bar{K},\bar{L}) = \mathcal{W}^s_+(K^i,L^i)$ is two-dimensional and $\mathcal{W}^s_-(\bar{K},\bar{L})$ is one–dimensional. □

The results above allow us to classify DGE paths according to two dimensions: (i) according to their crossing through $\mathcal{S}$ and (ii) according to the evolution of their determinacy properties over time. We call *regular DGE path* to a DGE path that does not cross $\mathcal{S}$ and *singular DGE path* to a DGE path does it. We say *determinacy is permanent* if the dimension of the stable manifold is the same throughout $\mathcal{W}^s(\bar{K},\bar{L})$, with the exception of an impasse point, and *determinacy is temporary* if the dimension of the stable manifold is not the same throughout $\mathcal{W}^s(\bar{K},\bar{L})$.

Proposition 4. *Let sets $\mathcal{S}$ be non-empty and assume there is one regular steady-state. Then, the following types of DGE paths are possible in the presence of impasse singularities:*

1. *Stationary DGE paths if the stable manifold associated to $(\bar{K},\bar{L})$ is empty and $K_0 = \bar{K}$.*
2. *Regular asymptotic stationary DGE paths, which are permanently determinate or indeterminate, and are confined to subset Ω_+ (Ω_-), if $(\bar{K},\bar{L}) \in \Omega_+$ (Ω_-) and $K_0 \in \mathcal{W}^s_+(\bar{K},\bar{L})$ $(\mathcal{W}^s_-(\bar{K},\bar{L}))$.*

3. *Singular asymptotic stationary DGE paths, which are not permanently determinate or indeterminate, if the stable manifold has different dimensions on subsets Ω_- and Ω_+, and if the initial point K_0 belongs to one branch of the stable manifold and the steady-state $(\bar{K},\bar{L})$ belongs to its complement. Two cases are possible: (1) the DGE path can be initially temporarily determinate and asymptotically indeterminate only if there is an impasse-transversal saddle point; or (2) the DGE path can be initially temporarily indeterminate and asymptotically determinate only if there is an impasse-transversal node.*

3 Singular macrodynamics

In this section we present two models with the structure of the benchmark model presented in section 2 to illustrate the two types of singularity. The first is the well known Benhabib and Farmer (1994) model and the second is a DGE model with a government following a cyclical fiscal policy rule.

3.1 The Benhabib and Farmer (1994) model

Benhabib and Farmer (1994) provide us a fine example of the slow-fast singularity. In this case, the production side is represented by $y\,(K,L) \equiv K^\alpha L^\beta$, $\alpha,\beta > 0$ with $\alpha+\beta > 1$, $r\,(K,L) \equiv aK^{\alpha-1}L^\beta$, and $w\,(K,L) \equiv bK^\alpha L^{\beta-1}$, where $a,b \in (0,1)$. Since the utility function is $u(C,L) = \log C - (1+\chi)^{-1} L^{1+\chi}$ with $\chi \geq 0$, we have $\theta\,(K,L) = 1$ and $v\,(C,K,L) \equiv -L^{\beta-1-\epsilon} + bC^{-1}K^\alpha L^{\beta-1}$, where $\epsilon\,(\varphi) \equiv \beta - (1+\chi)$ and $\varphi \equiv \begin{pmatrix} a & b & \alpha & \beta & \rho \end{pmatrix}^\top$.

To apply the results from subsection 2.3 we write the slow-fast system as

$$\begin{aligned} \dot{K} &= f_1(K,L,\epsilon) \equiv K^\alpha L^\beta (1 - bL^{\epsilon-\beta})\ , \\ \epsilon \dot{L} &= f_2^s(K,L,\epsilon) \equiv L\left[K^{\alpha-1}L^\beta\left(a - \alpha(1-bL^{\epsilon-\beta})\right) - \rho\right], \end{aligned}$$

and

$$\begin{aligned} K' &= \epsilon f_1\,(K,L,\epsilon) \equiv \epsilon K^\alpha L^\beta\left(1 - bL^{\epsilon-\beta}\right)\ , \\ L' &= f_2^s\,(K,L,\epsilon) \equiv L\left(K^{\alpha-1}L^\beta\left(a - \alpha\left(1 - bL^{\epsilon-\beta}\right)\right) - \rho\right)\ , \end{aligned}$$

where it is clear that a slow-fast singularity exists for $\epsilon = 0$ and there are no impasse singularities. There is a unique positive steady-state $(\bar{K},\bar{L})$ where $\bar{K} = (a/\rho)^{\frac{1}{1-\alpha}}\,\bar{L}^{\frac{\beta}{1-\alpha}}$ and $\bar{L} = b^{\frac{1}{\beta-\epsilon}}$. The Jacobian for the slow system has trace and determinant given by

$$\text{tr}DF(\bar{K},\bar{L}) = \frac{\rho^2(1-\alpha)(\beta-\epsilon)}{a\epsilon},\ \det DF(\bar{K},\bar{L}) = \frac{\rho\,(\beta(a-\alpha)+\epsilon\alpha)}{a\epsilon}.$$

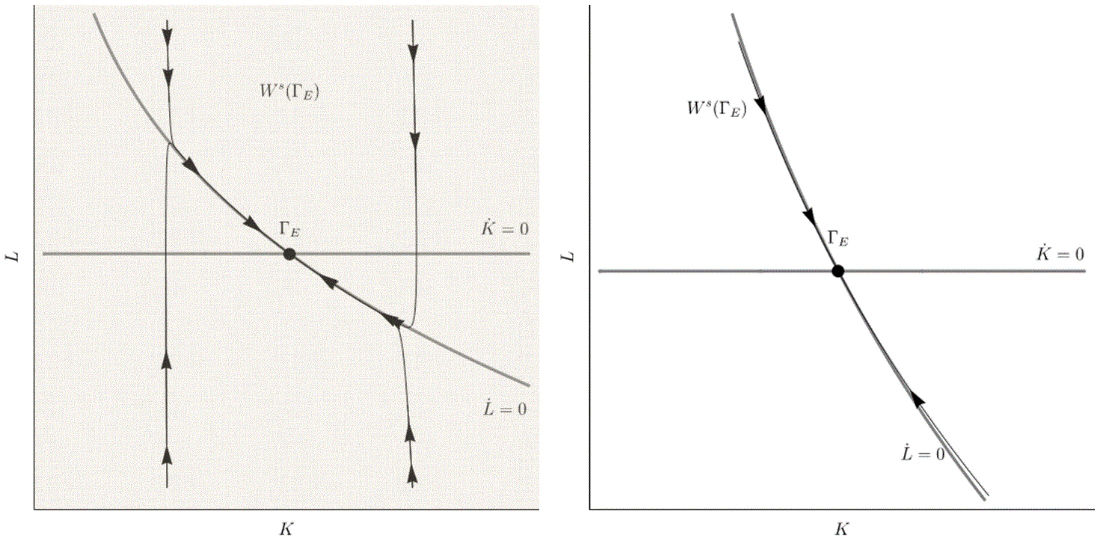

Figure 1: Phase diagrams for the Benhabib and Farmer (1994) for $\epsilon > 0$ (LHS panel) and $\epsilon < 0$ (RHS panel)

We can easily see that both these quantities can take infinite values for $\epsilon = 0$.

Henceforth, we use the authors' assumptions on the parameter values, i.e. we suppose that $\alpha > a$ and $\beta > b$. Thus, we find that the dimension of the stable manifold depends on ϵ: for $\epsilon < 0$ the steady-state is either a stable focus or node and for $\epsilon > 0$ it is a saddle[11].

Using the theory presented in subsection 2.3 we can draw further conclusions. First, the slow-fast subset (defined by $f_2(K, L, 0) = 0$) is $\mathcal{S}^p = \{(K, L) : L = h(K)\}$, where $h(K) \equiv \left(\frac{\rho K^{1-\alpha} - \alpha b}{a - \alpha}\right)^{\frac{1}{\beta}}$. Second, all points belonging to set $\mathcal{S}^p$ are slow-fast regular and attracting due to the fact that $f_{2,L}(K, L_p) = \beta(a - \alpha)K^{\alpha-1}L_p^{\beta-1} < 0$ and there are no singular slow-fast points.

Figure 1 illustrates the dynamic behavior of the Benhabib and Farmer (1994) model in the (K, L) space. On the left-hand-side (LHS) panel, we represent the phase diagram for $\epsilon > 0$, but small. We can observe that there is a unique steady-state equilibrium represented by point Γ_E. Since there are two negative eigenvalues associated with this stationary point, all DGE paths converge asymptotically to Γ_E. However, the steady-state is locally and

[11]Benhabib and Farmer (1994, p.34) already noted this behavior for particular values of the parameters: "As χ moves below -0.015 the roots both become real but remain negative until at (approximately) $\chi = -0.05$ [i.e. $\epsilon = 0$] one root passes through minus infinity and reemerges as a positive real root.". To compare with our results please note that we introduced a slight change in notation: while the authors set χ as non-positive we set χ as non-negative.

globally indeterminate, as there is an infinite number of initial points for a given K_0, leading to the long-run equilibrium. We can also see that L adjusts very fast so that the trajectory quickly approaches the isocline $\dot{L} = 0$ and then K starts adjusting more slowly until the steady-state is reached.

On the right-hand-side (RHS) panel, we represent the phase diagram for $\epsilon < 0$, also small. Now, the unique steady-state is locally and globally determinate, as there is one positive and one negative eigenvalue associated with it. For each initial level for the capital stock, K_0, there is only one value of L, such that convergence to the steady-state is asymptotically assured and the transversality condition holds. Notice that the stable manifold associated to the steady-state, $\mathcal{W}^s(\Gamma_E)$, stays very close to the $\dot{L} = 0$ isocline, meaning that labor adjusts faster than capital, as in the case $\epsilon > 0$.

For $\epsilon = 0$, we obtain a degenerate case where the adjustment of L is automatic, so that the stable manifold coincides with the $\dot{L} = 0$ isocline, and this curve is the geometrical analog of set $\mathcal{S}^p$.

3.2 Singularities generated by cyclical fiscal policy rules

In this section we present a DGE model for an economy with a government that follows a cyclical fiscal policy rule [12]. The production side is represented by a production function with constant returns to scale, i.e. $y(K, L) = K^{\alpha} L^{1-\alpha}$, with $\alpha \in (0, 1)$. Thus, we obtain $r(K, L) \equiv \alpha(L/K)^{1-\alpha}$, and $w(K, L) \equiv \beta(K/L)^{\alpha}$. The utility function is the same as in Benhabib and Farmer (1994).

The Government imposes a distortionary income tax on households with a flat rate $T \in (0, 1)$. Thus, the relevant input prices for households are $(1 - T)\, r(K, L)$ and $(1 - T)\, w(K, L)$. The tax revenue, $Ty(K, L)$, is used *within each period* to finance government final spending (G)[13].

We assume *that the distortionary tax rule takes* the form $T(K, L) = \phi\, y(K, L)^{\mu}$, where $\mu < 0$, i.e. the rule is countercyclical, such that the constraint $0 < T(K, L) < 1$ holds[14].

From the assumptions above and considering equation (3), we obtain the consumption function $C = c(L, K) \equiv (1 - T(K, L))w(K, L)L^{-\chi}$, which is a non-monotonic function of L,

[12] For another example in a macro model with imperfect competition see Brito, Costa, and Dixon (2016).

[13] Considering that we have an infinitely-lived representative household, it would act as if budget was balanced at all moments in time, i.e. Ricardian equivalence holds.

[14] One special case is given by $\phi = G$ and $\mu = -1$, corresponding to setting the expenditure level.

as $c_L \lesseqgtr 0$ if and only if $T(K,L) \lesseqgtr T_s \in (0,1)$ where

$$T_s \equiv \frac{\alpha+\chi}{\epsilon}\ ,\ \epsilon(\varphi) \equiv \alpha+\chi-(1-\alpha)\mu\ , \tag{30}$$

where $\varphi \equiv \begin{pmatrix}\alpha & \chi & \phi & \mu\end{pmatrix}^\top \in \Phi = (0,1)\times\mathbb{R}^3_{++}$ and $\epsilon > 0$ for all parameter values, thus preventing the existence of slow-fast singularities. The system (8) takes the form:

$$\begin{aligned} \dot{K} &= f_1(K,L) \equiv (1-T(K,L))\, y(K,L)\,(1-\ell(L))\ ,\\ \delta(K,L)\dot{L} &= f_2(K,L) \equiv L(1-T(K,L))\,(R(K,L)-\rho)\ , \end{aligned} \tag{31}$$

where

$$\delta(K,L) \equiv \epsilon(T(K,L)-T_s)\ , \tag{32}$$

$$R(K,L) \equiv r(K,L)\,(\mu T(K,L) + (1-(1+\mu)T(K,L))\ell(L))\ , \tag{33}$$

and $\ell(L) = (L^*/L)^{1+\chi}$, for $L^* = (1-\alpha)^{1+\chi}$. The domain for the two variables is

$$\Omega = \left\{(K,L) \in \mathbb{R}^2_{++} : T(K,L) < 1\right\}\ .$$

If we write the dynamic system as

$$\begin{aligned} \dot{K} &= (1-T(K,L))y(K,L)(1-\ell(L))\ ,\\ \dot{L} &= \frac{L(1-T(K,L))\,(R(K,L)-\rho)}{\epsilon\delta(K,L)}\ , \end{aligned} \tag{34}$$

it is clear that impasse-type singularities exist for values of (K,L) such that $\delta(K,L) = 0$, i.e. for $T(K,L) = T_s$.

The manifolds $T(K,L) = 1$ and $T(K,L) = T_s$ both limit the set of admissible values for (K,L), which is partitioned by the impasse set $\mathcal{S} = \{(K,L) : T(K,L) = T_s\}$ into a set of low tax rates $\Omega_- \equiv \{(K,L) : 0 < T(K,L) < T_s\}$ and high tax rates $\Omega_+ \equiv \{(K,L) : T_s < T(K,L) < 1\}$. Notice that, for a given L the high (low) tax rate set corresponds to lower (higher) values for K.

The steady-states $(\bar{K},\bar{L})$ of system (34) cannot be determined explicitly. However, we always have $\bar{L} = L^*$, then $\bar{\ell} = \ell(\bar{L}) = 1$ and $\bar{K} \in \left\{K : (1-T(K,\bar{L}))r(K,\bar{L}) = \rho\right\}$. Since $\rho > 0$, then the steady-state constraint $\bar{T} = T(\bar{K},\bar{L}) < 1$ always holds. Let us define a critical value for ϕ:

$$\phi^* \equiv T^* \left[\left(\frac{\alpha(1-T^*)}{\rho}\right)^{\frac{\alpha}{1-\alpha}} \bar{L}\right]^{-\mu},\ \text{for } T^* \equiv \frac{1-\alpha+\alpha\mu}{1-\alpha}\ .$$

Thus, we have two possible cases:

1. For $\phi = \phi^*$, the steady-state is unique (there could be a local regular bifurcation), i.e. $\Gamma_E = \{(\bar{K}^*, \bar{L})\}$ with $\bar{K}^* \equiv \left(\frac{\alpha(1-T^*)}{\rho}\right)^{\frac{1}{1-\alpha}} \bar{L}$.

2. For $\phi < \phi^*$, two steady-states exist, i.e. $\Gamma_E = \{ (\bar{K}_L, \bar{L}), (\bar{K}_H, \bar{L})\}$ such that $\bar{K}_L < \bar{K}^* < \bar{K}_H$. In this case we obtain the following relations between the steady-state tax rates: $\bar{T}_H < T^* < \bar{T}_L$, for $\bar{T}_i = T(\bar{K}_i, \bar{L})$ with $i = H, L$.

The Jacobian of the reduced system, (8), evaluated at any steady-state $(\bar{K}, \bar{L})$, has the trace and determinant given by

$$\mathrm{tr}DF(\bar{K}, \bar{L}) = -\frac{\rho(\alpha+\chi)(1-(1+\mu)\bar{T})}{\epsilon(\bar{T}-T_s)},\ \det DF(\bar{K}, \bar{L}) = \frac{\rho^2(1+\chi)(\mu^*-\mu)(T^*-\bar{T})}{\epsilon(\bar{T}-T_s)},$$

where $\mu^* \equiv (1-\alpha)/\alpha$ and $T^* \equiv \mu^*/(\mu^*-\mu)$. Let us define another critical value for ϕ:

$$\phi_i \equiv T_s \left[(\ell_i)^{-\frac{1}{1+\chi}} \left(\frac{\alpha(\mu T_i + (1-(1+\mu)T_i)\ell_i}{\rho}\right)^{\frac{1}{1+\chi^*}} \bar{L}\right]^{-\mu}$$

where $\ell_i \equiv \dfrac{(1+\chi^*)(\alpha+\chi)}{(1+\chi)(\chi^*-\chi}$ and $1+\chi^* \equiv \dfrac{1-\alpha}{\alpha}$.

Now, consider the generic case $\phi < \phi^*$, for which two steady-states exist. Three generic cases and one non-generic case are possible, concerning the local dynamic properties:

1. For $\phi < \phi_s$, two saddle steady-state equilibria exist such that the associated tax rates are $\bar{T}_L < \min\{T^*, T_s\} < \max\{T^*, T_s\} < \bar{T}_H$. The low tax rate steady-state is located in set Ω_- and the high tax rate steady-state is in set Ω_+.

2. For $\chi > \chi^*$ and $\phi^i < \phi < \phi^*$, one saddle steady-state equilibrium exists at $\bar{T}_L$ and one unstable node or focus exists at $\bar{T}_H$. Both steady-states are located in the low tax rate subset Ω_-.

3. For $\chi < \chi^*$ and $\phi^i < \phi < \phi^*$, one saddle steady-state equilibrium exists at $\bar{T}_H$ and one stable node or focus exists at $\bar{T}_L$. Both steady-states are located in the high tax rate subset Ω_+.

4. For $\phi = \phi_i$, one impasse-singular steady-state equilibrium exists at $\bar{T}_L$ $(\bar{T}_H)$ if $\chi < \chi^*$ $(> \chi^*)$.

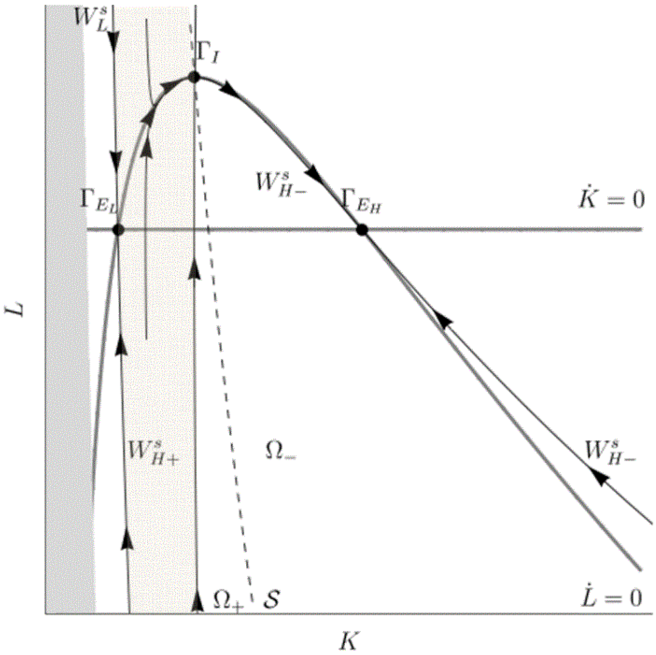

Figure 2: Phase diagram for the endogenous tax rule model

As far as impasse points (i.e. elements of set $\mathcal{S}$) are concerned, we have the following results. First, no impasse-tangent points exist, i.e. Γ_K is empty. Second, impasse-transversal points exist, i.e. $\Gamma_I = \{(K, L) \in \Omega : T(K, L) = T_s,\ R(K, L) = \rho\} \neq \varnothing$. Set Γ_I partitions set $\mathcal{S}$ into two (not necessarily compact) subsets: (i) the set of impasse-repeller points $\mathcal{S}_+ = \{(K, L) \in \mathcal{S} : R(K, L) < \rho\}$ and (ii) the set of impasse-atractor points $\mathcal{S}_- = \{(K, L) \in \mathcal{S} : R(K, L) < \rho\}$. In addition, Γ_I has only one element for $\chi > \chi^*$ and it has two elements for $\chi < \chi^*$ and $\phi < \phi_i$.

Considering that we can obtain two steady-states and two impasse-transversal points, this model goes beyond the conditions stated in section 2.4. In order to illustrate a type of DGE paths that can occur in models with singularities, but do not occur in regular models, Figure 2 presents a phase diagram illustrating a case for which $\chi < \chi^*$. In this case, two saddle steady-states (labelled H and L) exist and two impasse-transversal points (one shown in Figure 2, as Γ_I) exist. steady-state $(\bar{K}_L, \bar{L})$ is located in subset Ω_+, steady-state $(\bar{K}_H, \bar{L})$ is located in subset Ω_- and the stable manifolds associated with both are represented by $\mathcal{W}^s_L$ and $\mathcal{W}^s_H$. While $\mathcal{W}^s_L$ is located entirely on Ω_+, the stable manifold $\mathcal{W}^s_H$ is located in the union of two subsets together with the impasse-transversal node point Γ_I:

$\mathcal{W}^s_H \supset \mathcal{W}^s_{H+} \cup \mathcal{W}^s_{H-}$ where $\mathcal{W}^s_{H+}$ is two-dimensional and $\mathcal{W}^s_{H-}$ is one-dimensional. This means that the DGE path $(K_H(t), L(t))_{t\in[0,\infty}$ converging to the high tax rate steady-state displays *temporary indeterminacy*. That is, if $K_H(0) \in \mathcal{W}^s_{H+}$ there is an infinite number of values for $L_H(0)$ consistent with rational expectations equilibrium dynamics. All trajectories in this area converge to a point, at a finite moment in time t_s, depending upon $L_H(0)$, and after that moment converge through a common path to the steady-state $(\bar{K}_H, \bar{L})$.

Finally, notice that the stable manifold $\mathcal{W}^s_L$ defines the boundary to $\mathcal{W}^s_{H+}$. This clearly indicates that a local analysis using the usual approach, whilst appropriate for standard DGE models, is clearly misleading here.

4 Conclusion

4.1 A tale with illustrative metaphors

Before we conclude this article, the reader may find the usage of some metaphors helpful to understand the role of singularities in DGE models, especially that of impasse singularities[15]. For that purpose, we will make use of expressions used in astrophysics that became popular for the general audience, namely *black holes, white holes and wormholes*.

However, a word of caution is due. We do not intend to emulate gravitational fields here and we do not claim that there is an isomorphism from physical concepts to the ones used here. Despite the fact that mathematical structures behind physics models and those presented here are different, there is a strong analogy between the consequences of singularities in both types of model.

This is not new in economics. When Cass and Shell (1983) coined the phrase *sunspots* to represent random shocks that do not affect fundamentals, but end up affecting economic activity, they were not trying to emulate magnetic fields on the surface of stars. These are metaphors that help us understand the mathematics.

Attractor-impasse points, i.e. elements of set $\mathcal{S}_-$, can be seen as singularities within *black holes*, as trajectories within their basin of attraction cannot escape their strong "gravitational" pull. We saw that infinitely-living rational agents do not choose entering this area of space as they would eventually arrive at the singularity and remain there forever. Unless the singularity is an equilibrium a rational agent would never choose this path. Repeller-impasse points, i.e. elements of set $\mathcal{S}_+$, can be seen as singularities within *white holes*, as

[15]We thank an anonymous referee and Nuno Barradas for suggesting this clarification.

trajectories are drawn away from it. Impasse-transversal nodes work as a conduit between a *black-hole* and a *white-hole* singularity, thus linking different areas of space. Let us call them *wormholes*. Point Γ_I in Figure 2 provides such an example. It works like a *black hole* from the point of view of the (light) shaded area on the LHS of it. However, it also works like a *white hole* from the point of view of the area on its RHS of the impasse-transversal point, by expelling the trajectories onto a single determinate path towards Γ_{E_H}. Thus, it functions as a "portal" linking this otherwise two "parallel universes": one governed by high tax rates and another one by low taxes rates.

4.2 Final remarks

> The test of a good macro-model is not whether it predicts a little better in "normal" times, but whether it anticipates abnormal times and describes what happens then. Black holes "normally" don't occur. Standard economic methodology would therefore discard physics models in which they play a central role.
>
> In Stiglitz (2011, p.17)

Singularities play a large role in other scientific fields, including physics and electrical engineering (where they are related to systemic shut-downs). In economics, the phenomenon has remained largely unexplored or, in the spirit of Stiglitz's critique, just put aside as an "abnormal time." In this chapter we have hoped to rectify this "black hole" in economics.

Despite the simplicity of the DGE models we have looked at in section 3 of this paper, we have found singular dynamics were possible. There are also a large number of dynamic models with externalities, rules, and distortions which may well also give rise to singularities. We find it surprising that singular dynamics have been absent from the macroeconomic dynamics literature - a veritable "black hole" in macroeconomic theory.

In this paper we presented conditions for the emergence of singularities, described two types of singularities, slow-fast (perturbation) and impasse singularities, presented geometrical methods to deal with both of them, and applied our analysis to two simple cases, the Benhabib and Farmer (1994) model and a DGE model with a government following a cyclical fiscal policy rule. Because researchers have not known how to deal with singularities, we believe that they have either been ignored or avoided. We now have the tools for analyzing singularities and hope that this will mean that their implications within existing models can now be explored properly.

We hope that this article may provide a contribution to bring these *black holes* and the techniques associated with them to the mainstream discourse in economics. Strange as they may sound now, so did *sunspots* in the 1980s. Only the test of time can tell if this hope will materialise. Nonetheless, there is no shadow of a doubt over how Jean-Michel Grandmont's contributions to economics has stood that test of time.

References

Barnett, William A. and Susan He (2010). "Existence of singularity bifurcation in an Euler-equations model of the United States economy: Grandmont was right". In: *Economic Modelling* 27.6, pp. 1345 –1354.

Benhabib, J. and R. Farmer (1994). "Indeterminacy and increasing returns". In: *Journal of Economic Theory* 63, pp. 19–41.

Brito, Paulo, Luís Costa, and Huw Dixon (2016). "Singular macroeconomic dynamics and temporary indeterminacy". mimeo.

Cardin, Pedro Toniol, Paulo Ricardo da Silva, and Marco António Teixeira (2012). "Implicit differential equations with impasse singularities and singular perturbation problems". In: *Israel Journal of Mathematics* 189, pp. 307–322.

Cass, David and Karl Shell (1983). "Do sunspots matter ?" In: *Journal of Political Economy* 91.2, pp. 193–227.

Fenichel, N. (1979). "Geometric singular perturbation theory for ordinary differential equations". In: *Journal of Difference Equations and Applications* 31, pp. 53–98.

Grandmont, Jean-Michel (1985). "On endogenous competitive business cycles". In: *Econometrica* 53, pp. 995–1045.

— (1998). "Expectations formation and stability of large socioeconomic systems". In: *Econometrica* 66.4, pp. 741–782.

— (2008). "Nonlinear difference equations, bifurcations and chaos: An introduction". In: *Research in Economics* 62.3, pp. 122–177.

Grandmont, Jean-Michel, Patrick Pintus, and Robin de Vilder (1998). "Capital-Labor Substitution and Competitive Nonlinear Endogenous Business Cycles". In: *Journal of Economic Theory* 80.1, pp. 14–59.

Guckenheimer, John and Phillip Holmes (1990). *Nonlinear Oscillations and Bifurcations of Vector Fields*. 2nd. Springer-Verlag.

He, Yijun and William A. Barnett (2006). "Singularity bifurcations". In: *Journal of Macroeconomics* 28, pp. 5–22.

Kuehn, Christian (2015). *Multiple Time Scale Dynamics*. 1st ed. Applied Mathematical Sciences 191. Springer.

Kuznetsov, Yuri A. (2005). *Elements of Applied Bifurcation Theory*. 3rd. Springer.

Llibre, J., J. Sotomayor, and M. Zhitomirskii (2002). “Impasse bifurcations of constrained systems”. In: *Fields Institute Communications* 31, pp. 235–256.
Stiglitz, J. (2011). “Rethinking macroeconomics: what failed, and how to repair it”. In: *Journal of the European Economic Association* 9.4, pp. 591–645.
Zhitomirskii, M. (1993). “Local normal forms for constrained systems on 2-manifolds”. In: *Boletim da Sociedade Brasileira de Matemática* 24, pp. 211–232.

A Appendix

To simplify notation let $x = (x_1, x_2) \equiv (K, L)$ and consider functions $f_1(x)$, $f_2(x)$, and $\delta(x)$. In addition, consider the two vector fields $F(x)$, as in equation (13), and $F^r(x)$, as in equation (27). Then, at a regular steady-state $(\bar{x})$, we have $f_1(\bar{x}) = f_2(\bar{x}) = 0$ and $\delta(\bar{x}) \neq 0$. Furthermore, at a generic impasse-transversal point (x^i), we have $\delta(x^i) = f_2(x^i) = 0$ and $f_1(x^i) \neq 0$.

Lemma 4. *Assume there is one generic impasse-transversal point x^i and one regular steady-state $\bar{x}$, both belonging to a set X, such that $f_{2,x_2}(x)$ has the same sign for any point $x \in X$. Therefore,* $\operatorname{sign}(\det DF^r(x^i)) = -\operatorname{sign}(\det DF(\bar{x}))$.

Proof. For sake of simplicity, let us set $\epsilon(\varphi) = 1$. The determinants of the Jacobian for $F(x)$ evaluated at the steady-state and at an impasse-transversal point are respectively given by

$$\begin{aligned}\det DF(\bar{x}) &= \frac{f_{1,x_1}(\bar{x})\, f_{2,x_2}(\bar{x}) - f_{1,x_2}(\bar{x})\, f_{2,x_1}(\bar{x})}{\delta(\bar{x})},\\ \det DF^r(x^i) &= f_1(x^i)\left(\delta_{x_1}(x^i)\, f_{2,x_2}(x^i) - \delta_{x_2}(x^i)\, f_{2,x_1}(x^i)\right).\end{aligned}$$

First, note that both points share a common condition $f_2(x_1, x_2) = 0$. If this function is differentiable, we can write $\nabla f_2(x) \cdot dx = 0$. By computing Taylor approximations to $f_1(x^i)$ in a neighbourhood of $\bar{x}$ and to $\delta(\bar{x})$ in a neighbourhood of x^i, and considering the differentiability of $f_2(\cdot)$, we obtain:

$$\begin{aligned}f_1(x^i) &= \frac{\det DF(\bar{x})\delta(\bar{x})}{f_{2,x_2}(\bar{x})}(x_1^i - \bar{x}_1),\\ \delta(\bar{x}) &= \frac{\det DF^r(x^i)}{f_1(x^i) f_{2,x_2}(x^i)}(\bar{x}_1 - x_1^i).\end{aligned}$$

Thus,

$$\frac{\det DF^r(x^i)}{\det DF(\bar{x})} = -(\delta(\bar{x}))^2 \frac{f_{2,x_2}(x^i)}{f_{2,x_2}(\bar{x})}.$$

□