Cardiff Economics Working Papers

Working Paper No. E2016/13

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December 2016

ISSN 1749-6010

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Equilibria in a Japanese-English Auction with Discrete Bid Levels for the Wallet Game*

Ricardo Gonçalves† and Indrajit Ray‡

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Abstract

We consider the set-up of a Japanese-English auction with exogenously fixed discrete bid levels for the wallet game with two bidders. We prove that bidding twice the signal - the equilibrium strategy with continuous bid levels - is never an equilibrium in this set up. We show that partition equilibria exist that may be separating or pooling. We illustrate some separating and pooling equilibria with two and three discrete bid levels; we also compare the revenues of the seller from these equilibria and thereby find the optimal bid levels in these cases.

Keywords: Japanese-English auctions, wallet game, discrete bids, partitions, pooling equilibrium, separating equilibrium.

JEL Classification Numbers: C72, D44.

*We wish to thank seminar and conference participants at Aveiro, Bristol, Coimbra, Glasgow, Jadavpur, Lisbon, New Delhi, Warwick, York and in particular, Bhaskar Dutta, Francoise Forges, Peter Hammond, Ron Harstad, Paul Klemperer, Fabio Michelucci and Stephen Morris for many helpful comments, as well as Steve Farrar (Looe Fish Selling) for invaluable insights. Financial support from Fundação para a Ciência e Tecnologia (through project UID/GES/00731/2013) and the British Academy/Leverhulme small research grant (SG153381) is gratefully acknowledged.

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1 INTRODUCTION

Milgrom and Weber (1982) analysed a particular version of the English auction, the so-called Japanese-English Auction (henceforth JEA) in which the price of the object increases continuously and the bidders must keep on pressing a button whilst they are interested in buying the object at the posted price; the auction ends when all but one bidder release the button. Later, Klemperer (1998) focused on a particular common value auction with two bidders, popularly known as the "wallet game" (in which the common value is simply the sum of two private signals, the "wallets"), as a special case of the above model and illustrated that bidding twice the individual signal forms the unique symmetric (Bayesian-Nash) equilibrium in this game.

The set-up in this paper is same as the usual JEA, except that the price, goes up in discrete commonly known bid levels. As in the usual JEA, if a bidder wants to drop out, all he has to do is release the button. The final auction price is equal to the highest bid level at which at least one bidder was active. We use the wallet game (with two bidders) as our background game to theoretically analyse this JEA with discrete bid levels.

In the recent past, English auctions with predefined discrete bid levels have been analysed. In these studies, bidders have to choose among the exogenously fixed bid levels when it is their turn to bid (Rothkopf and Harstad, 1994; David *et al*, 2007) or at the very least, increase the going bid by a minimum increment (Isaac *et al*, 2007). Following the seminal experiment by Avery and Kagel (1997) on a JEA based on the wallet game, not surprisingly, there is now a growing theoretical and experimental literature on this issue (Yu, 1999; Sinha and Greenleaf, 2000; Cheng, 2004; Gonçalves and Hey, 2011). However, to the best of our knowledge, nobody so far has attempted to theoretically characterise the equilibria of a JEA in a common value environment with exogenously specified discrete bid levels. We take the first step to this direction in this short paper.

Discrete bidding in English auctions is the norm in the real world, although substantial variations in the exact characteristics of the auctions are observed.[1] In most English auctions, admittedly, the discrete bids are endogenous, possibly a function of several factors, including number of bidders, (expected) bidders' valuations, etc. Cassady (1967) gives examples of auctions in which the bid levels are known, such as tobacco and livestock auctions in the USA. In auctions at Sotheby's or Christie's, bidding usually advances between 5% and 10% of the current price level (Rothkopf and Harstad, 1994). Even in real world examples of Japanese-English auctions (commonly known as clock auctions), the price actually increases in discrete increments. For example, in the Looe wholesale fish auction (UK), the increments are anywhere from 1p to 5p or 10p and sometimes different increments are used for different

[1] Harstad and Rothkopf (2000) and Isaac *et al* (2007), for example, provide alternative and under some circumstances, perhaps more realistic, English auction models.

species during the same auction session.[2] Online auction sites, such as eBay, use variants of such English auctions, adapted to the online world (Bajari and Hortaçsu, 2004), where bid increments are also discrete (and depend on the price level).[3] Bidding at the online auction site[4] QXL was quite similar to our model: the price went up in predetermined increments and if bids were not a multiple of that increment, then the bid was rounded down to the closest multiple of the increment.[5]

In our game (JEA with discrete bid levels for the wallet game with two bidders), we first prove that one cannot construct a symmetric equilibrium using bids that are twice the private signal (as in the case of continuous bid levels illustrated by Klemperer, 1998). We then show that (symmetric) partition equilibria, involving weakly increasing strategies based on partitions of the signal space, exist for the wallet game in a JEA with discrete bid levels. Such partition equilibria may be *pooling* or *separating* (depending on the number of partitions). We illustrate several such equilibria with only two or three discrete bid levels (with certain parametric restrictions). These equilibria, however, yield a lower expected revenue for the seller than in the case of a continuous JEA. Despite this, we further show that a revenue-maximising second best solution for this set-up exists; that is, the seller may choose these bid levels optimally to maximise the revenue.

2 THE GAME

We consider the wallet game in which there are two symmetric risk-neutral bidders $i \in \{1,2\}$ who compete for the purchase of one single good, whose value, $\tilde{V}$, is common but *ex ante* unknown to both bidders. Each bidder receives an independent and uniformly distributed[6] private signal $x_i \sim U(0,1)$, $i = 1,2$. The (ex ante) unknown common value of the good is simply the sum of the two signals: $\tilde{V} = x_1 + x_2$.

We make use of the JEA with some exogenously fixed discrete bids. In our set up, as in the usual JEA, the price increases; however the bid levels are discrete (rather than continuous) and are fixed exogenously.

[2] In wholesale fish markets, ascending (electronic) auctions are commonly used (Graham, 1999, p. 181), as they replicate (electronically) the traditional oral ascending auctions; however, the descending (Dutch) auction is also used (Guillotreau and Jimenez-Toribio, 2011). Discrete but known bid increments are a common feature in both these auction types (Carleton, 2000, pp. 10-11).

[3] One example that sort of fits our model is eBay where it is not all that rare to specify that bids cannot start below a given price, say, \$49, and must be a multiple of \$1 with the last digit 9; however, the assumption of a commonly known upper limit does not hold there. We thank Ron Harstad for providing this example.

[4] The UK site closed down in 2008.

[5] QXL bidding increments depended on the bid value. For example, for bids in the £2.50 − £9.99 range, the bid increment was £0.10 while for bids in the £10 − £99.99, it was £1.00 and so on (as illustrated by the auction rules in www.qxl.com).

[6] The uniform distribution is undoubtedly easier to compute solutions for, however, any other specific distribution should not matter in our analysis.

Formally, the bid levels are the elements of the set $A = \{a_1, ..., a_k\}$, with $0 < a_1 < ... < a_k < 2$, $k \geq 2$ a finite integer; the set A is common knowledge to the bidders. We will denote a typical bid level by a_j, for $j = 1, ..., k$, with the implicit assumption that $a_0 = 0$ and $a_{k+1} = 2$, for notational convenience whenever required in this paper.

The commonly-known (publicly displayed) auction price goes up in discrete bid levels in the set A starting from a_1 and ending at a_k. The bidders have to keep pressing a button at each bid level to be actively bidding; if a bidder wants to drop out of the auction at any stage, all he has to do is release the button. The final auction price is equal to the highest bid level in which at least one bidder was active. This rule implies that, for any $j = 1, ..., k-1$, if one bidder is active at a_j but not at a_{j+1} while his opponent is active at a_{j+1}, then the latter wins the auction and pays a price equal to a_{j+1}; by contrast, if both bidders are active at a_j, but not at a_{j+1}, then the auction winner is decided at random with equal probabilities and the final price is a_j; finally, if both bidders are active at the last bid level a_k, the winner will be chosen at random with equal probabilities and will pay the price a_k. The net payoff to the (selected) winner in each of the above cases is the realised value of $x_1 + x_2$ minus the price to pay while the payoff to the loser is 0. If no bidder is active at a_1, then the auction ends immediately and the payoff to either bidder is 0.

A strategy in this Bayesian game is therefore to choose (as in the standard JEA) a drop out bid level as a function of the signal. Given a signal $x \in (0, 1)$, a bidding strategy for a player thus chooses 0 (which implies that the bidder is not active even at a_1) or a bid level a_j so that the bidder will be active at a_j but not at a_{j+1}, where $j = 1, ..., k$ (with $a_{k+1} = 2$). We denote a typical strategy by σ which is a function dented by $b(x) \in \{0, a_1, ..., a_k\}$, which implies that the player with signal x is active until $b(x)$.

The JEA for the wallet game with k bid levels $(a_1, .., a_k)$ as described above will henceforth be called G_k. Let us now look at possible strategies of G_k.

Definition 1 *A strategy $\sigma = b(x)$ for G_k is weakly increasing (decreasing) if for all pair of signals x and y, $x > y$, $b(x) \geq (\leq)\ b(y)$.*

Theoretically, there are strategies that are neither weakly increasing nor weakly decreasing. For example, consider a strategy σ^{rat} for which $b(x) = a_m$, when x is a rational number and $b(x) = a_n$, otherwise for some m and n.

Understandably, bidders may not wish to use the strategy 0. Formally,

Definition 2 *A strategy is called active if it never chooses* 0 *for any signal, i.e., the bidder is active at least at a_1 for any signal x. A strategy is called inactive if it chooses* 0 *for at least one signal, i.e., the bidder is inactive even at a_1 for some signal x.*

A natural type of strategy one may think of is a strategy that divides the domain of the signal x, the interval $(0, 1)$, into $(l + 1)$ subintervals or partitions using l (≥ 1) many cut-off signals.

Definition 3 *A partition strategy for G_k is a strategy that uses l (≥ 1) cut-off points and thus $(l+1)$ partitions of the interval $(0,1)$, and chooses an element from the set $\{0, a_1, ..., a_k\}$ for each of these partitions.*

Note that $l = 0$, which implies no cut-off signal and therefore no partition, also generates a feasible strategy; in such a strategy, only one bid level is picked for the whole set of signals, the interval $(0,1)$.

Definition 4 *In G_k, a strategy is called a babbling strategy, if regardless of the signal, the bidder chooses either 0 or a particular bid level a_j, $j = 1, ..., k$. In an active babbling strategy, the bidder chooses a particular bid level a_j, $j = 1, ..., k$, regardless of the signal. An inactive babbling strategy chooses 0 for any signal x.*

Obviously, there are strategies that are not partition strategies; for example, the above mentioned σ^{rat} is not a partition strategy. Also, a partition strategy may be neither weakly increasing nor weakly decreasing. For example, consider G_2 with two bid levels, L and H and think of a strategy written using two cut-offs x^* and y^* as:

$$\sigma = \begin{cases} L \text{ if } x \leq x^* \\ H \text{ if } x^* < x \leq y^* \\ L \text{ if } x > y^* \end{cases}$$

We now focus on a specific subset of the strategy sets in G_k and make the following assumption.

Assumption 0. All the bidders use weakly increasing partition strategies only.

A weakly increasing strategy in G_k^0 can be written in terms of some cut-off signals x_c^*, $c = 1, ..., l$, where $0 < x_1^* < ... < x_l^* < 1$ and $l \leq k$ that divide the interval $(0,1)$ into $(l+1)$ partitions and associates an element of $\{0, a_1, ..., a_k\}$ to each partition in an increasing order.

The JEA for the wallet game with k bid levels $(a_1, .., a_k)$ with weakly increasing partition strategies only is our baseline game and we henceforth call it G_k^0. In any G_k^0, a non-babbling strategy, σ, can be easily associated with a certain probability distribution over the set $\{0, a_1, ..., a_k\}$, as determined by the partition(s). A babbling strategy is clearly associated with a degenerate distribution (probability 1 on one element of the set $\{0, a_1, ..., a_k\}$).

In the following section we find possible equilibria of the game G_k^0, with $k > 2$, using the standard notion of Bayesian-Nash equilibrium with usual expected payoffs.

3 RESULTS

We focus only on symmetric equilibria for the game G_k^0, with $k > 2$, in the rest of our paper. As it is well-known, the symmetric (Bayesian-Nash) equilibrium for the JEA with continuous bids is given by bid functions $b_i^*(x_i) = 2x_i$, $i = 1, 2$, as derived by Milgrom and Weber (1982), in a general model, and later specifically for the wallet game by Klemperer (1998) and Avery and Kagel (1997). An immediate question, therefore, is whether these equilibrium strategies also form an equilibrium in the JEA with discrete bids for the wallet game or not.

A direct translation of the above (continuous) JEA bidding strategies into our setting would yield the following bidding strategy: each bidder i should stay active in the auction until the bid reaches $b_i^*(x_i) = 2x_i$ and drop out after that. The associated bidding strategy for each bidder therefore is a weakly increasing partition strategy with cut-offs $\frac{a_j}{2}$, $j = 1, ..., k$, i.e., each bidder i will choose 0 if $2x_i < a_1$ (equivalent to $x_i < \frac{a_1}{2}$) and will choose $a_j \in A$ (active at bid level a_j but would drop out at bid level a_{j+1}) if $a_j \leq 2x_i < a_{j+1}$ (equivalent to $\frac{a_j}{2} \leq x_i < \frac{a_{j+1}}{2}$), for $j = 1, ..., k-1$ and will choose a_k if $2x_i > a_k$ (equivalent to $x_i > \frac{a_k}{2}$). Let us call this partition strategy "twice-signal bidding". We show that this bidding strategy in our setting is not an equilibrium.

Proposition 1 *The twice-signal bidding strategy profile is not an equilibrium in* G_k^0.

Proof. We will prove Proposition 1 by showing that there are signal realisations for which there exists some profitable individual deviation for a bidder. We will illustrate this using bidder 1's strategy. Without loss of generality, suppose $x_1 > x_2$ where x_1 and x_2 are the signals of two respective bidders; further assume that for some j, $j = 1, ..., k$, $a_{j-1} \leq 2x_2 < a_j \leq 2x_1 < a_{j+1}$ (assume $a_0 = 0$ and $a_{k+1} = 2$ if required). Hence, following the twice-signal bidding strategy, bidder 2 would be active at bid level a_{j-1} but would drop out at bid level a_j while bidder 1 would be active at a_j - the price that bidder 1, the winning bidder, would pay.

Note that bidder 1's expected payoff, conditional on winning at a_j, is given by:
$\pi_1 = x_1 + E\left[X_2 \middle| \frac{a_{j-1}}{2} \leq X_2 < \frac{a_j}{2}\right] - \frac{a_j}{2} = x_1 + \frac{\frac{a_{j-1}}{2} + \frac{a_j}{2}}{2} - a_j$ which is equal to $x_1 + \frac{a_{j-1}}{4} - \frac{3}{4}a_j$.

Now suppose, bidder 1's signal realisation is 'too low' within the chosen interval $\frac{a_j}{2} \leq x_1 < \frac{a_{j+1}}{2}$, that is $x_1 = \frac{a_j}{2} + \varepsilon$ for some small $\varepsilon > 0$. In this case, he will find the expected value of the good to be lower than a_j, thus yielding negative profits. This is because in such a case, bidder 1's expected payoff π_1 will be $\frac{a_j}{2} + \varepsilon + \frac{a_{j-1}}{4} - \frac{3}{4}a_j = \frac{a_{j-1} - a_j}{4} + \varepsilon < 0$, for an appropriately chosen small ε. ■

Note that the above proof of Proposition 1 goes through even in G_k as the deviation does not require a partition strategy. Thus, we have the following.

Corollary 1 *The twice-signal bidding strategy profile is not an equilibrium in* G_k.

Clearly, there is a discontinuity in the fact that the twice-signal bidding strategy is an equilibrium in the continuous case but not so in the discrete case. We prove this formally.

Corollary 2 *The twice-signal bidding strategy profile is not an equilibrium in* G_k*, for any finite* k*, but is an equilibrium in the corresponding game with continuous bids.*

Proof. The proof follows immediately from the proof of Proposition 1. Consider any natural number k and thus a JEA with k many bid levels. Take the minimum of the distances between two successive bid levels and call it δ; hence, for any j, $a_j - a_{j-1} \geq \delta$. One can now choose a small enough ε such that $\varepsilon < \delta/4$. Now, in the case mentioned in last line of the proof of Proposition 1 above, bidder 1's expected payoff ($\pi_1 = \frac{a_{j-1}-a_j}{4} + \varepsilon$) will definitely be strictly negative. Hence, the twice-signal bidding strategy profile is not an equilibrium in the JEA with k many discrete bids, for any finite k, while as we already know (Klemperer, 1998), twice-signal bidding does constitute an equilibrium in the continuous case. ■

Although twice-signal bidding is not an equilibrium G_k^0, we will show that other equilibria exist for our game in the next subsection.

3.1 Partition Equilibria

In this subsection, we characterise different equilibria using partition strategies. We further focus on active partition strategies to find equilibria in G_k^0, with $k > 2$. Clearly, using Definitions 1, 2 and 3, for any active weakly increasing partition strategy the number of cut-offs l must be $\leq k-1$.

We call an active weakly increasing partition strategy *separating* if $l = k-1$, for any $k \geq 2$. Clearly, for $k = 2$, an active weakly increasing partition strategy is either babbling or separating with a single cut-off. For $k > 2$, we call a partition strategy *pooling* if $1 \leq l < k-1$.

Definition 5 *In G_k^0, where $k > 2$, a separating strategy is an active weakly increasing partition strategy that uses $k-1$ cut-offs $(x_1^*, ..., x_{k-1}^*)$ and thereby k partitions; it can be written as:*

$$\sigma = \begin{cases} a_1 \text{ if } x \leq x_1^* \\ a_j \text{ if } x_{j-1}^* < x \leq x_j^*,\ j = 2, ..., k-1 \\ a_k \text{ if } x > x_{k-1}^* \end{cases}$$

In G_2^0, with 2 bid levels (a_1, a_2) and one cut-off x^*, a separating strategy σ can be written as: $\sigma = a_1$ if $x \leq x^*$ and a_2 otherwise.[7]

Similarly, one may also formally define and express any *pooling strategy*, in G_k^0, with $k > 2$, using l ($< k-1$) cut-offs.

As mentioned earlier, a non-babbling partition strategy, σ, can be interpreted as a probability distribution. For example, for $k > 2$, the separating strategy in Definition 5 above is a strategy in which the bidder chooses a_1 with probability x_1^*, a_j with probability $(x_j^* - x_{j-1}^*)$, $j = 2, ..., k-1$ and a_k with probability $\left(1 - \sum_{j=1}^{k-1} x_j^*\right)$. The probabilities for a pooling strategy can also be similarly identified.

[7] In this definition, we have used, without any loss of generality, the weak inequality on the left hand side of the cut-off (as the signal is generated using a continuous distribution). One may define a partition strategy with the weak inequality on the right hand side of the cut-off in which case the following equilibrium analysis needs to be modified accordingly.

We may now define a *partition equilibrium*, using the above partition strategies. As mentioned earlier, we are going to consider symmetric equilibria only. An equilibrium in symmetric partition (babbling) strategies is a strategy profile in which both bidders play the same partition (babbling) strategy.

A symmetric separating (pooling) partition equilibrium can be characterised by a separating (pooling) strategy with usual (Bayesian-Nash) equilibrium conditions. The equilibrium conditions are: *i.* indifference at the cut-offs, *ii.* incentive constraints for each partition, *iii.* activation constraint (active at a_1) which implies the participation constraint (at the beginning of the auction) and *iv.* feasibility constraints for the cut-off points. One can thus define and characterise such a partition equilibrium using these conditions.

Definition 6 *In G_k^0, a symmetric strategy profile (σ_1, σ_2) is called a separating equilibrium if each bidder i uses the same separating strategy σ_i with $k-1$ cut-offs $(x_1^*, ..., x_{k-1}^*)$ with all of the following conditions satisfied.*[8]

$u_1(a_j, \sigma_2)|_{x_1=x_j^*} = u_1(a_{j+1}, \sigma_2)|_{x_1=x_j^*}$, $j = 1, ..., k-1$ *[indifference conditions]*

$u_1(a_1, \sigma_2) > u_1(a_h, \sigma_2)$ *if* $x_1 \leq x_1^*$, $h > 1$ *[incentive constraint for the first partition]*

$u_1(a_k, \sigma_2) > u_1(a_h, \sigma_2)$ *if* $x_1 > x_{k-1}^*$, $h < k$ *[incentive constraint for the last partition]*

$u_1(a_j, \sigma_2) > u_1(a_h, \sigma_2)$ *if* $x_{j-1}^* < x_1 \leq x_j^*$, $j = 2, ..., k-1$, $h \neq j$ *[incentive constraints for all other partitions, needed only for $k > 2$]*

$u_1(a_1, \sigma_2) \geq u_1(0, \sigma_2) = 0$ *if* $x_1 \leq x_1^*$ *[activation constraint] implying* $u_1(a_1, \sigma_2)|_{x_1=0} \geq 0$ *[participation constraint]*

$0 < x_1^* < ... < x_{k-1}^* < 1$ *[feasibility constraints]*

Similarly, one may write down the equilibrium conditions for a (symmetric) pooling partition equilibrium or even a (symmetric) babbling equilibrium. The conditions for a babbling equilibrium clearly involve just the incentive constraint and the participation constraint.

Unfortunately, it is extremely difficult to analytically solve the above set of constraints (as in Definition 6) and thereby find all partition equilibria for G_k^0, particularly when k is not small. The analysis is understandably easier for G_2^0 or G_3^0. In the next subsection, we will consider G_2^0 and G_3^0 and show examples of symmetric partition equilibria in such games.

3.2 Separating Equilibrium in G_2^0

Consider any given G_2^0; let us denote the bid levels by L (low) and H (high); that is, $k = 2$ with $a_1 = L$ and $a_2 = H$. Further, we make the following assumption on the values of L and H.

Assumption 1. $L < \frac{1}{2}$ and $L + \frac{1}{2} < H < \frac{3}{4} + \frac{L}{2}$.

[8] Abusing notations for the expected payoff from a partition strategy.

Note that Assumption 1 in turn implies $H < 1$.

Any separating strategy here can be written in terms of a cut-off signal x^*; a separating strategy for some x^*, $0 < x^* < 1$, is thus:

$$\sigma^{2S} = \begin{cases} L \text{ if } x \leq x^* \\ H \text{ if } x > x^* \end{cases}$$

In a symmetric separating equilibrium, each bidder thus plays L with probability x^* (the probability that $x \leq x^*$) and H with probability $(1 - x^*)$, that is, the strategy σ^{2S} can be associated with the distribution $(x^*; 1 - x^*)$ over L and H.

We are now ready to present the separating equilibrium of this game.

Proposition 2 *Under Assumption 1, the separating strategy $\sigma^{2S} = (x^*; 1 - x^*)$, with $x^* = \frac{2H-1}{2(1+L-H)}$, constitutes a symmetric separating equilibrium of G_2^0.*

Proof. We first compute the (expected) payoffs for a bidder from the partition strategy profile; without loss of generality, we consider bidder 1. When bidder 2 has a signal $x_2 \leq x^*$ and bids L, using the uniform distribution, bidder 1 expects bidder 2 to have a signal realisation equal to $x^*/2$; similarly, when bidder 2 has a signal $x_2 > x^*$ and bids H, bidder 1 expects bidder 2 to have a signal realisation equal to $(1 + x^*)/2$.

Bidder 1's expected payoffs thus are given by: $u_1 \left(L, \sigma^{2S}\right) = x^*.\frac{1}{2}(x_1 + \frac{x^*}{2} - L) + (1 - x^*).0$ and $u_1 \left(H, \sigma^{2S}\right) = x^*.(x_1 + \frac{x^*}{2} - H) + (1 - x^*).\frac{1}{2}.(x_1 + \frac{1+x^*}{2} - H)$.

Setting the indifference condition (as in Definition 6) $u_1 \left(L, \sigma^{2S}\right) = u_1 \left(H, \sigma^{2S}\right)$, we get $x^* = \frac{2x_1+1-2H}{2(H-L)}$, which implies that when $x_1 = x^*$, $u_1 \left(L, \sigma^{2S}\right) = u_1 \left(H, \sigma^{2S}\right)$ provided $x^* = \frac{2H-1}{2(1+L-H)}$.

Substituting this cut-off x^* in the expected payoffs, we obtain

$u_1 \left(L, \sigma^{2S}\right) - u_1 \left(H, \sigma^{2S}\right) = \frac{1}{4}\frac{2H-1-2x_1(1+L-H)}{1+L-H} = \frac{1}{2}(x^* - x_1)$.

Hence, for bidder 1, if $x_1 > x^*$, we have $u_1 \left(H, \sigma^{2S}\right) > u_1 \left(L, \sigma^{2S}\right)$, that is, with a high signal realisation (above x^*), bidder 1 prefers to bid H, and when $x_1 \leq x^*$, we have $u_1 \left(L, \sigma^{2S}\right) > u_1 \left(H, \sigma^{2S}\right)$, that is, with a low signal realisation (below x^*), bidder 1 prefers to bid L, which confirms the desired equilibrium condition (incentive constraint as in Definition 6).

We now have to confirm the feasibility constraint that $x^* \in (0, 1)$; this is guaranteed by Assumption 1 as $x^* > 0 \Leftrightarrow H > 1/2$ and $x^* < 1 \Leftrightarrow H < \frac{3}{4} + \frac{L}{2}$.

Finally, we need to check the activation (and thus the participation) constraint that the payoffs cannot be negative (otherwise bidders would prefer not to be active) at L. As $u_1 \left(L, \sigma^{2S}\right)$ is increasing in x_1, we just need to ensure that $u_1 \left(L, \sigma^{2S}\right)\big|_{x_1=0} = \frac{(1-2H)(1+2L)(2L+1-H)}{16(H-L-1)^2} > 0$.

The above is indeed true; the denominator is always positive and for the numerator to be positive we must have either $H < 1/2$ and $H < L + 1/2$, which we disregard because it would not yield a positive cut-off x^*, or we must have $H > 1/2$ and $H > L + 1/2$, which is guaranteed under Assumption 1. ■

It is also easy to show that the above partition equilibrium is indeed unique (in weakly increasing symmetric strategies). Clearly, there are only two potential candidate profiles which are based on two babbling strategies of staying active until L or H regardless of the signal. We denote these profiles by (L, L) and (H, H) respectively and prove that neither of them is an equilibrium.

Corollary 3 *Under Assumption* 1*, the separating strategy profile* $(\sigma^{2S}, \sigma^{2S})$*, where,* $\sigma^{2S} = (x^*; 1 - x^*)$*, with* $x^* = \frac{2H-1}{2(1+L-H)}$*, is the unique symmetric (Bayesian-Nash) equilibrium of* G_2^0*.*

Proof. To show uniqueness, we just need to prove that (L, L) and (H, H) cannot be an equilibrium. To prove that (L, L) cannot be an equilibrium, we note that there are realisations of x_1 for bidder 1 for which bidding L is not a best response against L. To see this, take $1 > x_1 > 1 - 2(\frac{3}{4} + \frac{L}{2} - H)$. In this case, $u_1(H, L) - u_1(L, L) = (x_1 + \frac{1}{2} - H) - \frac{1}{2}(x_1 + \frac{1}{2} - L) > 0$ (as, by Assumption 1, $1 - 2(\frac{3}{4} + \frac{L}{2} - H) < 1$). Similarly, we prove that (H, H) cannot be an equilibrium by showing that there are realisations of x_1 for bidder 1 for which bidding H is not a best response against H. To see this, take $0 < x_1 < H - 1/2$. Here, $u_1(L, H) - u_1(H, H) = \frac{1}{2}(H - \frac{1}{2} - x_1) > 0$. ■

The above results thus fully characterises the equilibrium of G_2^0s under Assumption 1, as the following example shows.

Example 1 *Consider two specific values for L and H, namely,* $L = 1/5$ *and* $H = 4/5$*, satisfying Assumption* 1*. In this case, from Proposition* 2*, we have* $x^* = 3/4$*. Hence, in the unique symmetric equilibrium of this game, a bidder is active at L (but not at H) if and only if the signal is less than or equal to* 3/4*. Bidder i's payoff,* u_i*, from this equilibrium strategy profile is given by* $u_i = \frac{3}{8}x_i + \frac{21}{320}$ *if* $x_i \leq 3/4$ *(in which case bidder i plays L) and* $u_i = \frac{7}{8}x_i - \frac{99}{320}$ *if* $x_i > 3/4$ *(in which case bidder i plays H).*

3.3 Pooling Equilibria with Three Bid Levels

Now we consider G_3^0 to provide some examples of pooling equilibria. Let us denote three bid levels by L (low), M (medium) and H (high); that is, $k = 3$ with $a_1 = L$, $a_2 = M$ and $a_3 = H$. We illustrate three different types of pooling equilibria with three bid levels in the following subsections.

3.3.1 Illustration 1

In this illustration, we use the parameter values from the previous subsection (G_2^0) and extend it to a specific G_3^0. We take any values of L and H satisfying Assumption 1 and call them L and M respectively (Assumption 1′ below) and make a further assumption (Assumption 2) on H as below.

Assumption 1′. $L < \frac{1}{2}$ and $L + \frac{1}{2} < M < \frac{3}{4} + \frac{L}{2}$.

Assumption 2. $H > \frac{3}{4} + \frac{M}{2} + \frac{2M-1}{8(1+L-M)}$.

Clearly, Assumption $1'$ is same as Assumption 1 with renamed parameters. We now construct a pooling equilibrium using the same cut-off as in Proposition 2. Let us consider the following partition strategy:

$$\sigma^{3P_1} = \begin{cases} L \text{ if } x \leq x^* \\ M \text{ if } x > x^* \end{cases}$$

Clearly the above strategy is a pooling strategy as the bid level H is not used. In a symmetric profile, each bidder plays L with probability x^* and M with probability $(1 - x^*)$, that is, the strategy σ^{3P_1} can be associated with the distribution $(x^*; 1 - x^*; 0)$ over L, M and H. We now prove that this strategy profile is an equilibrium for this game (following the proof of Proposition 2).

Proposition 3 *Under Assumptions* $1'$ *and* 2, *the partition strategy* $\sigma^{3P_1} = (x^*; 1 - x^*; 0)$, *with* $x^* = \frac{2M-1}{2(1+L-M)}$, *constitutes a symmetric pooling equilibrium of* G_3^0.

Proof. We first compute bidder 1's expected payoffs under this partition strategy profile which turns out to be:

$u_1\left(L, \sigma^{3P_1}\right) = x^* \frac{1}{2}\left(x_1 + \frac{x^*}{2} - L\right)$; $u_1\left(M, \sigma^{3P_1}\right) = x^*\left(x_1 + \frac{x^*}{2} - M\right) + (1 - x^*)\frac{1}{2}\left(x_1 + \frac{1+x^*}{2} - M\right)$.

The indifference condition (as in Definition 6), $u_1\left(L, \sigma^{3P_1}\right) = u_1\left(M, \sigma^{3P_1}\right)$, is satisfied provided $x^* = \frac{2M-1}{2(1+L-M)}$.

Using this cut-off, we obtain $u_1\left(L, \sigma^{3P_1}\right) - u_1\left(M, \sigma^{3P_1}\right) = \frac{1}{2}(x^* - x_1)$; therefore the incentive constraints $u_1\left(L, \sigma^{3P_1}\right) > u_1\left(M, \sigma^{3P_1}\right)$ if $x_1 < x^*$ (and thus the constraint $u_1\left(L, \sigma^{3P_1}\right) > u_1\left(H, \sigma^{3P_1}\right)$ if $x_1 < x^*$) and $u_1\left(M, \sigma^{3P_1}\right) > u_1\left(L, \sigma^{3P_1}\right)$ if $x_1 > x^*$ are all satisfied.

Hence, we just need to prove that bidder 1 does not deviate and play H when $x_1 > x^*$, that is, we must have $u_1\left(H, \sigma^{3P_1}\right) - u_1\left(M, \sigma^{3P_1}\right) < 0$ if $x_1 > x^*$. Note that $u_1\left(H, \sigma^{3P_1}\right) - u_1\left(M, \sigma^{3P_1}\right) = \frac{1}{2}x_1 + \frac{1+x^*}{4} - H + \frac{M}{2}$. Substituting the value of x^* and setting $x_1 = 1$ (the highest possible signal), we confirm that this payoff difference is indeed negative under Assumption 2 (that is, $H > \frac{3}{4} + \frac{M}{2} + \frac{2M-1}{8(1+L-M)}$).

Finally, using the proof of Proposition 2, here as well we have the feasibility constraint and the activation (thus participation) constraint satisfied. ■

To illustrate the above, we may use the values in Example 1.

Example 2 *Take* $L = 1/5$, $M = 4/5$ *and* $H = 7/5$, *satisfying Assumptions* $1'$ *and* 2. *As in Example* 1, *here as well, we have* $x^* = 3/4$. *Thus in this symmetric pooling equilibrium of this game, a bidder is active at* L *(but not at* M *or* H*) when the signal is less than or equal to* 3/4 *and active at* M *(but not at* H*) when the signal is bigger than* 3/4. *Bidder* i*'s payoff,* u_i*, from this equilibrium strategy profile is given by* $u_i = \frac{3}{8}x_i + \frac{21}{320}$ *if* $x_i \leq 3/4$ *(in which case bidder* i *plays* L*) and* $u_i = \frac{7}{8}x_i - \frac{99}{320}$ *if* $x_i > 3/4$ *(in which case bidder* i *plays* M*).*

3.3.2 Illustration 2

In this illustration, we will use different parameter values to illustrate another pooling equilibrium for any given G_3^0; we make the following assumptions.

Assumption 1$''$. $L < \frac{1}{2}$ and $2L < M < \frac{3}{4} + \frac{L}{2}$.

Assumption 3. $H = \frac{1}{2} + 2M - L$.

Let us consider the following partition strategy:

$$\sigma^{3P_2} = \begin{cases} L \text{ if } x \leq x^* \\ H \text{ if } x > x^* \end{cases}$$

In this pooling strategy the bid level M is not used. Here, the strategy σ^{3P_2} can be associated with the distribution $(x^*; 0; 1 - x^*)$ over L, M and H. We now prove our next result.

Proposition 4 *Under Assumptions* $1''$ *and* 3*, the partition strategy* $\sigma^{3P_2} = (x^*; 0; 1 - x^*)$*, with* $x^* = \frac{4}{3}M - \frac{2}{3}L$*, constitutes a symmetric pooling equilibrium of* G_3^0.

Proof. Following Definition 6, we need to show that the equilibrium conditions are satisfied at these parameter values.

The indifference condition is met when $x^* = \frac{4}{3}M - \frac{2}{3}L$ as $u_1\left(L, \sigma^{3P_2}\right)\big|_{x_1 = x^*} = u_1\left(H, \sigma^{3P_2}\right)\big|_{x_1 = x^*} = \frac{2(2M-L)(M-L)}{3}$.

The activation (and thus participation) constraint is satisfied by Assumption $1''$ as $u_1\left(L, \sigma^{3P_2}\right)\big|_{x_1 = 0} = \frac{2(2M-L)(M-2L)}{9} \geq 0$ when $M > 2L$.

Note that the feasibility constraint $0 < x^* = \frac{4}{3}M - \frac{2}{3}L < 1$ is satisfied under Assumption $1''$.

We now need to prove the incentive constraints for the two partitions below and above x^*.

To do this, take a small $\varepsilon > 0$ and x_1 such that $|x_1 - x^*| = \varepsilon$. It is easy to check that at x_1, $u_1\left(L, \sigma^{3P_2}\right) - u_1\left(H, \sigma^{3P_2}\right)$ is $\frac{\varepsilon}{2} > 0$, when $x_1 < x^*$ and is $-\frac{\varepsilon}{2} < 0$, when $x_1 > x^*$. Similarly, at x_1, $u_1\left(L, \sigma^{3P_2}\right) - u_1\left(M, \sigma^{3P_2}\right)$ is $\frac{(2M-L)\varepsilon}{3} > 0$, when $x_1 < x^*$ and is $\frac{(L-2M)\varepsilon}{3} < 0$, when $x_1 > x^*$ (by Assumption $1''$). Finally, when $x_1 > x^*$, at x_1, $u_1\left(H, \sigma^{3P_2}\right) - u_1\left(M, \sigma^{3P_2}\right) = \frac{(3-4M+2L)}{6}\varepsilon > 0$ (by Assumption $1''$). Thus all the incentive constraints are satisfied. ■

We may illustrate the above result now using some specific parameter values.

Example 3 *Take* $L = 1/5$*,* $M = 3/5$ *and* $H = 3/2$*, satisfying Assumptions* $1''$ *and* 3*. From Proposition 4, we have* $x^* = 2/3$*. Thus in this symmetric pooling equilibrium of this game, a bidder is active at L (but not at* M *or* H*) when the signal is less than or equal to* $2/3$ *and active at* H *when the signal is bigger than* $2/3$*. Bidder i's payoff,* u_i*, from this equilibrium strategy profile is given by* $u_i = \frac{1}{3}x_i + \frac{2}{45}$ *if* $x_i \leq 2/3$ *(in which case bidder i plays L) and* $u_i = \frac{5}{6}x_i - \frac{13}{45}$ *if* $x_i > 2/3$ *(in which case bidder i plays* H*).*

3.3.3 Illustration 3

In this illustration, we make the following assumptions on the parameters.

Assumption 4. $M < \frac{1}{2}$.

Assumption 5. $H = M + \frac{1}{2}$.

Let us now consider the following partition strategy:

$$\sigma^{3P_3} = \begin{cases} M \text{ if } x \leq x^* \\ H \text{ if } x > x^* \end{cases}$$

In this pooling strategy the bid level L is not used. We may write the above strategy as $\sigma^{3P_3} = (0; x^*; 1 - x^*)$. We now prove that this strategy constitutes a symmetric equilibrium for this game.

Proposition 5 *Under Assumptions* 4 *and* 5*, the partition strategy* $\sigma^{3P_3} = (0; x^*; 1-x^*)$*, with* $x^* = 2M$*, constitutes a symmetric pooling partition equilibrium of* G_3^0*.*

Proof. Following Definition 6, we need to show that the equilibrium conditions are satisfied at these parameter values.

The indifference condition is satisfied at $x^* = 2M$, as $u_1\left(M, \sigma^{3P_3}\right)\big|_{x_1=x^*} = u_1\left(H, \sigma^{3P_3}\right)\big|_{x_1=x^*} = M^2$. The activation (and thus participation) constraint is trivially satisfied as $u_1\left(M, \sigma^{3P_3}\right)\big|_{x_1=0} = 0$. The feasibility constraint $0 < x^* = 2M < 1$ is met by Assumption 4.

We now need to prove the incentive constraints for the two partitions below and above x^*. To do this, as in the proof of Proposition 4, we take a small $\varepsilon > 0$ and x_1 such that $|x_1 - x^*| = \varepsilon$. It is easy to check that at x_1, $u_1\left(M, \sigma^{3P_3}\right) - u_1\left(H, \sigma^{3P_3}\right)$ is $\frac{\varepsilon}{2} > 0$, when $x_1 < x^*$ and is $-\frac{\varepsilon}{2} < 0$, when $x_1 > x^*$. Similarly, whenever $x_1 < x^*$, at x_1, $u_1\left(L, \sigma^{3P_3}\right) - u_1\left(M, \sigma^{3P_3}\right) = -M\left(2M + \varepsilon\right) < 0$. Finally, whenever $x_1 > x^*$, at x_1, $u_1\left(L, \sigma^{3P_3}\right) - u_1\left(H, \sigma^{3P_3}\right) = -2M^2 - M\varepsilon - \frac{1}{2}\varepsilon < 0$. Thus all the incentive constraints are satisfied. ∎

We may now illustrate the above result.

Example 4 *Take* $L = 1/10$*,* $M = 2/5$ *and* $H = 9/10$*, satisfying Assumptions* 4 *and* 5*. From Proposition* 5*, we have* $x^* = 4/5$*. Thus in this symmetric pooling equilibrium of this game, a bidder is active at* M *(but not at* H*) when the signal is less than or equal to* 4/5 *and active at* H *when the signal is bigger than* 4/5*. Bidder* i*'s payoff,* u_i*, from this equilibrium strategy profile is given by* $u_i = \frac{2}{5}x_i$ *if* $x_i \leq 4/5$ *(in which case bidder* i *plays* M*) and* $u_i = \frac{9}{10}x_i - \frac{2}{5}$ *if* $x_i > 4/5$ *(in which case bidder* i *plays* H*).*

3.4 Multiple Pooling Equilibria

In this subsection, we show that there may exist two pooling equilibria in a given G_3^0 (for given values of the bid levels), using the illustrations in the previous subsection.

It is clear that one cannot find values of three bid levels so that both pooling equilibria $(\sigma^{3P_1}, \sigma^{3P_1})$ and $(\sigma^{3P_3}, \sigma^{3P_3})$ exist simultaneously (as Assumptions 1 and 4 for values of M are mutually exclusive). Similarly, one cannot find values of three bid levels for which both pooling equilibria $(\sigma^{3P_2}, \sigma^{3P_2})$ and $(\sigma^{3P_3}, \sigma^{3P_3})$ exist (as both Assumptions 3 and 5 cannot be satisfied by the same value of H).

However it is possible to find values of the bid levels such that both pooling equilibria $(\sigma^{3P_1}, \sigma^{3P_1})$ and $(\sigma^{3P_2}, \sigma^{3P_2})$ exist simultaneously.

Note that any values of L and M satisfying Assumption $1^{'}$ will also satisfy Assumption $1^{''}$ as for any $L < \frac{1}{2}$, $M > L + \frac{1}{2}$ implies $M > 2L$. Hence, we may find a set of numerical values for three bid levels for which two pooling equilibria exist as the following example (similar to Example 2) illustrates.

Example 5 *Take a G_3^0 with $L = 1/5$, $M = 4/5$ and $H = 19/10$, satisfying Assumption $1^{'}$ (and thereby Assumption $1^{''}$) and Assumptions 2 and 3. In this game, we have two different pooling equilibria, $(\sigma^{3P_1}, \sigma^{3P_1})$ and $(\sigma^{3P_2}, \sigma^{3P_2})$, characterised by two different cut-offs, respectively, $3/4$ and $14/15$. First, the symmetric pooling partition equilibria, $(\sigma^{3P_1}, \sigma^{3P_1})$ exists (as in Example 2) in which each bidder is active at L (but not at M or H) when the signal is less than or equal to $3/4$ and active at M when the signal is bigger than $3/4$. Bidder i's payoff, u_i, from this equilibrium strategy profile is given by $u_i = \frac{3}{8}x_i + \frac{21}{320}$ if $x_i \leq 3/4$ (in which case bidder i plays L) and $u_i = \frac{7}{8}x_i - \frac{99}{320}$ if $x_i > 3/4$ (in which case bidder i plays M). Second, the symmetric pooling partition equilibria, $(\sigma^{3P_2}, \sigma^{3P_2})$ exists in which each bidder is active at L (but not at M or H) when the signal is less than or equal to $14/15$ and active at H when the signal is bigger than $14/15$. Bidder i's payoff, u_i, from this equilibrium strategy profile is given by by $u_i = \frac{7}{15}x_i + \frac{28}{225}$ if $x_i \leq 14/15$ (in which case bidder i plays L) and $u_i = \frac{29}{30}x_i - \frac{77}{225}$ if $x_i > 14/15$ (in which case bidder i plays H). One may compare these two equilibria by their ex-ante expected payoffs (for each bidder i) that are respectively $\frac{43}{160}$ $(= 0.26875)$ for $(\sigma^{3P_1}, \sigma^{3P_1})$ and $\frac{323}{900}$ $(= 0.35889)$ for $(\sigma^{3P_2}, \sigma^{3P_2})$; hence, the equilibrium σ^{3P_2} is better for the bidders.*

3.5 Seller's Expected Revenue

We now focus on the seller's expected revenue from all the equilibria stated in the previous subsections.

3.5.1 Revenue in G_2^0

Consider the separating equilibrium $(\sigma^{2S}, \sigma^{2S})$ as presented in Proposition 2. The expected revenue for the seller from this equilibrium is given by L when both players play L (occurs with probability $(x^*)^2$) and H in all other cases (i.e., when at least one bidder bids H). Thus the seller's expected revenue (R^{2S}) is: $R^{2S} = (x^*)(x^*)L + (x^*)(1 - x^*)H + (1 - x^*)(x^*)H + (1 - x^*)(1 - x^*)H = \frac{L+4LH-4LH^2+3H-4H^2+4HL^2}{4(1+L-H)^2}$.

We observe that for all values of L and H satisfying our assumption, the seller's expected revenue is lower than in a JEA with continuous bids, $E\left[P^{JEA}\right] = 2/3$ (see Avery and Kagel, 1997). The following figure (Figure 1) displays this result, which is similar to that obtained by Rothkopf and Harstad (1994, Proposition, p. 575) in a private values setting (insofar as the revenue from a discrete bidding auction is lower than in its continuous counterpart).

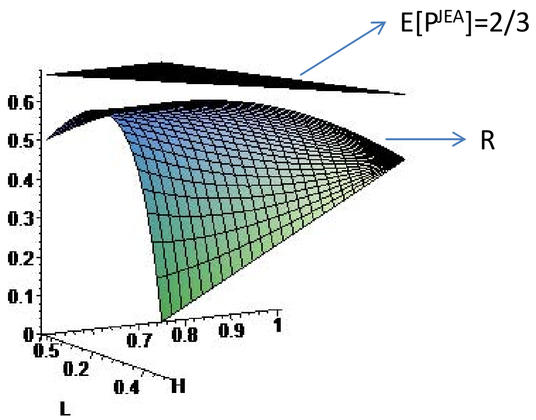

Figure 1: Seller's expected revenue for different bid levels

Although G_2^0 yields 'lost revenue' compared to the continuous case, it is possible to show that a second-best solution for the choice of L and H exists in this set-up.

Proposition 6 *In the equilibrium* $(\sigma^{2S}, \sigma^{2S})$ *as stated in Proposition* 2*, seller's expected revenue is maximised when* $L^* = 1/4$ *and* $H^* = 3/4$*, yielding* $x^* = 1/2$ *and* $R^{2S^*} = 5/8$*.*

Proof. In order to obtain the revenue-maximising values of L and H, we need to solve the following optimisation problem (rearranging the inequality restrictions):

$\max_{L,H} R^{2S} = \frac{L+4LH-4LH^2+3H-4H^2+4HL^2}{4(1+L-H)^2}$

subject to $1/2 - L \geq 0$, $H - L - 1/2 \geq 0$, $3/4 + L/2 - H \geq 0$, $L \geq 0$ and $H \geq 0$.

We set up the Lagrangian as below, where y_i are the multipliers:

$Z = \frac{L+4LH-4LH^2+3H-4H^2+4HL^2}{4(1+L-H)^2} + y_1 (1/2 - L) + y_2 (H - L - 1/2) + y_3 (3/4 + L/2 - H)$

We are now going to use the Kuhn-Tucker conditions for the above Langrangean. First, as we are looking for $L^* > 0$ and $H^* > 0$, we have $\frac{\partial Z}{\partial L} = 0$ and $\frac{\partial Z}{\partial H} = 0$. Now, when $\frac{\partial Z}{\partial y_2} = H - L - 1/2 = 0$ (that is, when $H = L + 1/2$), we have $y_2 > 0$ and the expected revenue is a concave function of L. This implies $\frac{\partial Z}{\partial y_1} = 1/2 - L > 0$ and also $\frac{\partial Z}{\partial y_3} = 3/4 + L/2 - H > 0$, thereby $y_1 = 0$ and $y_3 = 0$.

Thus we have three equations, namely, $\frac{\partial Z}{\partial L} = 0$, $\frac{\partial Z}{\partial H} = 0$ and $\frac{\partial Z}{\partial y_2} = 0$ that we can solve with respect to L, H and y_2. Solving these, we get $L^* = 1/4$ and $H^* = 3/4$ (with $y_2^* = 3/4$). For these optimal bid levels, $R^{2S^*} = 5/8$. ■

In the second best solution, the 'loss of revenue' compared to the JEA with continuous bids is approximately 6.3%. It is, although significantly higher than zero, not very high in percentage terms.

3.5.2 Revenue in G_3^0

We now consider the seller's revenue for each of the three pooling equilibria for any given G_3^0 as described above. For each case, we find the best parameter values that maximise the corresponding seller's revenue.

First we consider the pooling equilibrium $(\sigma^{3P_1}, \sigma^{3P_1})$ for G_3^0 which is very similar to the separating equilibrium $(\sigma^{2S}, \sigma^{2S})$ for G_2^0. The seller's revenue from the equilibrium $(\sigma^{3P_1}, \sigma^{3P_1})$ is given by:

$R^{3P_1} = \frac{L+4LM-4LM^2+3M-4M^2+4ML^2}{4(1+L-M)^2}$.

It is obvious that we will have the same values for the parameters that maximise the seller's revenue here.

Corollary 4 *Seller's expected revenue from the equilibrium $(\sigma^{3P_1}, \sigma^{3P_1})$ is maximised when $L^* = 1/4$, $M^* = 3/4$ and $H^* = 5/4$, yielding $x^* = 1/2$ and $R^{3P_1^*} = 5/8$.*

Proof. The proof follows immediately from Proposition 6. Given the solutions of the Lagrangean (as in the proof of Proposition 6) $L^* = 1/4$ and $M^* = 3/4$, we obtain $H^* = \frac{3}{4} + \frac{M^*}{2} + \frac{2M^*-1}{8(1+L^*-M^*)} = 5/4$. For these bid levels, $x^* = 1/2$ and $R^{3P_1^*} = 5/8$. ■

Note that, not surprisingly, $R^{3P_1^*} = R^{2S^*}$.

We now consider the pooling equilibrium $(\sigma^{3P_2}, \sigma^{3P_2})$. The seller's revenue from the equilibrium $(\sigma^{3P_2}, \sigma^{3P_2})$ is given by:

$R^{3P_2} = \frac{28}{9}LM - \frac{16}{9}M^2 - \frac{1}{3}L + \frac{1}{2} + \frac{2}{3}M - \frac{10}{9}L^2$.

Proposition 7 *Seller's expected revenue from the equilibrium $(\sigma^{3P_2}, \sigma^{3P_2})$ is maximised when $L^* = 1/4$, $M^* = 1/2$ and $H^* = 5/4$, yielding $x^* = 1/2$ and $R^{3P_2^*} = 5/8$.*

Proof. The proof is similar to that of Proposition 6. Given the constrained maximisation problem, we write down the corresponding Lagrangean and use the Kuhn-Tucker conditions. Solving, we get $L^* = 1/4$ and $M^* = 1/2$. Hence, $H^* = \frac{1}{2} + 2M^* - L^* = 5/4$. For these bid levels, $x^* = \frac{4}{3}M^* - \frac{2}{3}L^* = 1/2$ and therefore $R^{3P_2^*} = 5/8$. ■

Finally, we consider the pooling equilibrium $(\sigma^{3P_3}, \sigma^{3P_3})$. The seller's revenue from the equilibrium $(\sigma^{3P_3}, \sigma^{3P_3})$ is given by:

$R^{3P_3} = M - 2M^2 + \frac{1}{2}$.

Proposition 8 *Seller's expected revenue from the equilibrium* $(\sigma^{3P_3}, \sigma^{3P_3})$ *is maximised at* $M^* = 1/4$ *and* $H^* = 3/4$*, with any* $L < 1/4$*, yielding* $x^* = 1/2$ *and* $R^{3P_3^*} = 5/8$*.*

Proof. The proof is straightforward. From the first order condition, we obtain $M^* = 1/4$, in which case $H^* = M^* + 1/2 = 3/4$. Any $L < M^* = 1/4$ will thus be revenue-maximizing. In this case, $x^* = 1/2$ and $R^{3P_3^*} = 5/8$. ■

Observe that $R^{3P_1^*} = R^{3P_2^*} = R^{3P_3^*} = R^{2S^*}$. It is not really surprising if we carefully look at the way the pooling strategies σ^{3P_1}, σ^{3P_2} and σ^{3P_3} have been constructed as extreme points of a separating equilibrium in a G_3^0 (discussed in the next subsection) and hence the corresponding equilibrium profiles $(\sigma^{3P_1}, \sigma^{3P_1})$, $(\sigma^{3P_2}, \sigma^{3P_2})$ and $(\sigma^{3P_3}, \sigma^{3P_3})$ have the same payoffs.

3.6 Separating Equilibrium in G_3^0: A Simulation

One may be interested in constructing a separating equilibrium for any given G_3^0. Following Definition 5, a separating strategy for G_3^0 with three bid levels, L, M and H can be written using two cut-offs x^* $(= x_1^*)$ and y^* $(= x_2^*)$ as:

$$\sigma^{3S} = \begin{cases} L \text{ if } x \leq x^* \\ M \text{ if } x^* < x \leq y^* \\ H \text{ if } x > y^* \end{cases}$$

From Definition 6, we can construct a symmetric separating equilibrium using the above strategy. The profile $(\sigma^{3S}, \sigma^{3S})$ is an equilibrium if the following conditions are met.

$u_1\left(L, \sigma^{3S}\right)\big|_{x_1 = x^*} = u_1\left(M, \sigma^{3S}\right)\big|_{x_1 = x^*}$ [indifference at x^*]

$u_1\left(M, \sigma^{3S}\right)\big|_{x_1 = y^*} = u_1\left(H, \sigma^{3S}\right)\big|_{x_1 = y^*}$ [indifference at y^*]

$u_1\left(L, \sigma^{3S}\right) > u_1\left(M, \sigma^{3S}\right)$ if $x_1 < x^*$ [incentive constraint for the first partition]

$u_1\left(M, \sigma^{3S}\right) > u_1\left(L, \sigma^{3S}\right)$ if $x^* < x_1 < y^*$ [first incentive constraint for the second partition]

$u_1\left(M, \sigma^{3S}\right) > u_1\left(H, \sigma^{3S}\right)$ if $x^* < x_1 < y^*$ [second incentive constraint for the second partition]

$u_1\left(H, \sigma^{3S}\right) > u_1\left(M, \sigma^{3S}\right)$ if $x_1 > y^*$ [incentive constraint for the third partition]

$u_1\left(L, \sigma^{3S}\right) \geq u_1\left(0, \sigma_2\right) = 0$ if $x_1 \leq x^*$ [activation constraint] implying $u_1\left(L, \sigma^{3S}\right)\big|_{x_1 = 0} \geq 0$ [participation constraint]

$0 < x^* < y^* < 1$ [feasibility constraint]

As mentioned earlier, it is difficult to analytically characterise such an equilibrium, that is, it is hard to find numerical values for x^* and y^* satisfying all the above constraints for any given values of L, M and H. We thus present a simulation to indicate the existence of such an equilibrium for a fixed set of values of L, M and H. We start off with $L = 1/4$ and $M = 3/4$; recall that these values maximise the seller's revenue from the equilibrium $(\sigma^{2S}, \sigma^{2S})$ with two bid levels. Coupled with these values, we take a range of values for H between $5/4$ $(= 1.25)$ and $7/4$ $(= 1.75)$. Note that, for the bid levels $L = 1/4$,

$M = 3/4$, and $H = 5/4$, we have the pooling equilibrium $(\sigma^{3P_1}, \sigma^{3P_1})$ and for $L = 1/4$, $M = 3/4$, and $H = 7/4$, we have the pooling equilibrium $(\sigma^{3P_2}, \sigma^{3P_2})$. We vary the value of H and find values of x^* and y^* satisfying all the equilibrium conditions and thereby find a separating equilibrium in this case. The following figure (Figure 2) shows the cutoffs x^* and y^* in the separating equilibrium for different values of H (between 1.25 and 1.75 on the horizontal axis).

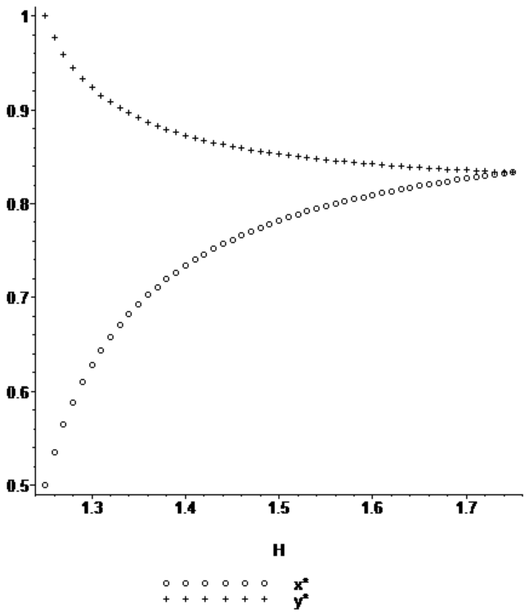

Figure 2: Cutoffs for separating equilibrium

In Figure 2, for each value of H (on the horizontal axis) we have two different dots: the lower curve is for x^* while the upper curve is for y^*. Take, for example, three different levels of H: $H = 7/5$, $H = 3/2$ and $H = 8/5$. The approximate numerical values are the following:

	$H = 7/5$	$H = 3/2$	$H = 8/5$
x^*	0.734	0.782	0.801
y^*	0.873	0.853	0.842
Seller's Revenue	0.514	0.476	0.453

Note that at the two boundaries of the values of H, we have the pooling equilibria $(\sigma^{3P_1}, \sigma^{3P_1})$ and $(\sigma^{3P_2}, \sigma^{3P_2})$ that can be interpreted as the two extremes of the separating equilibrium. The pooling equilibrium $(\sigma^{3P_1}, \sigma^{3P_1})$ is equivalent to a separating equilibrium with $x^* = 1/2$ and $y^* = 1$ in which H is not played. Similarly, the pooling equilibrium $(\sigma^{3P_2}, \sigma^{3P_2})$ is equivalent to a separating equilibrium with $x^* = y^* = 5/6$ in which M is not played.

We can find the seller's revenue from such a separating equilibrium, as displayed in Figure 3.

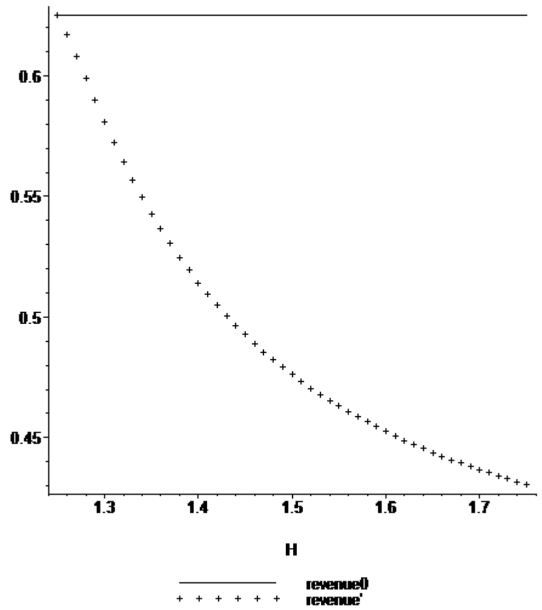

Figure 3: Seller's revenue in pooling and separating equilibrium

We observe that the revenue from any separating equilibrium here (revenue′ in Figure 3) is lower than that of the pooling equilibrium $(\sigma^{3P_1}, \sigma^{3P_1})$ which is equivalent to the equilibrium $(\sigma^{2S}, \sigma^{2S})$ with two bid levels (revenue0 in Figure 3).

Thus, we note that in this example, the seller strictly prefers the pooling equilibrium $(\sigma^{3P_1}, \sigma^{3P_1})$ to be played rather than the separating equilibrium for any sufficiently high H where these two types of equilibria coexist. Also, by the same token, we observe that two bid levels are (weakly) better than three for the seller. However, it is important to note that if the seller had the choice of all three bid levels, $L = 1/4$, $M = 3/4$ and $H > 5/4$ would in all likelihood not be his revenue-maximising choices. But finding the optimal choice of L, M and H is not easy, even with simulations. We conjecture that perhaps the revenue from the equilibrium $(\sigma^{2S}, \sigma^{2S})$ in G_2^0 is (weakly) higher than that from any (pooling or separating) equilibrium in G_3^0.

4 CONCLUSION

We have shown that the standard equilibrium (of bidding twice the signal) in JEA with continuous bid levels is not an equilibrium in a setting where bid levels are discrete; nevertheless, a partition equilibrium based on cut-offs in signals exists in a modified game where the bidders use only weakly increasing partition strategies. We have characterised these equilibria that can be pooling or separating. We illustrated a few such equilibria with two and three discrete bid levels. Under our partition equilibrium, seller's expected revenue is strictly lower than that of the continuous JEA; the seller can, however, optimally choose the bid levels to maximise the expected revenue. In this second best solution, the 'loss of revenue' compared to the JEA with continuous bid increments is not very high in percentage terms. Our paper thus provides some understanding of how, once one fixes the number of bid levels, bid levels should be optimally chosen by the seller.

The rationale behind our result is relatively straightforward; given discrete bid levels, the partition equilibrium leads players to bid up to the lowest discrete bid level 'too' often, and that reduces the expected revenue compared to the continuous bidding JEA. With continuous bid levels the players can easily infer (from the equilibrium strategies) their opponent's signal and thus accurately calculate their payoff, however, with discrete bid levels such an accurate inference is no longer possible and bidding up to the low bid level more often provides a 'safety net' under such "uncertainty".

Our construction of equilibrium is somewhat similar to the recent work by Ettinger and Michelucci (2016) and Hernando-Veciana and Michelucci (2016) in a different environment: these results are all related to a type of bunching which is somehow endogenously determined (in their papers, by jump bids or by the choice of a 2-stage mechanism while in our work by the choice of the bid levels).

Needless to add, whether a general result for the set of equilibria can be obtained for more than three bid levels is certainly an interesting question; future research should characterise the set of all such partition equilibria for any number of discrete bids and other (non-partition) equilibria, if any.

JEA with discrete bids may present other advantages to the auctioneer or to the bidders, such as, reduced auction duration or an easier understanding of the rules. Thus, it may very well be the case that it becomes a more attractive auction format in the future, in which case more analysis should be devoted to this format than its continuous bid counterpart.

Our research points out what the implications are of using a specific set of bid levels and how a seller should optimally manipulate it. One may be interested in finding the optimal number of bid levels for such an auction. Our simulation on three bid levels suggests that the optimal number of bid levels (to maximise the seller's revenue) is perhaps small. One may also be interested in testing this hypothesis in a suitably designed experiment. These are likely to be the next steps in our research.

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