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Incumbency and Entry in License Auctions: The Anglo-Dutch Auction Meets Other Simple Alternatives*

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Abstract

The existence of ex-ante strong incumbents may constitute a barrier to entry in auctions for goods such as licenses. Introducing inefficiencies that favor entrants is a way to induce entry and thus create competition. Designs such as the Anglo-Dutch auction have been proposed with this goal in mind. We first show that indeed the Anglo-Dutch auction fosters entry and increases the revenues of the seller. However, we argue that a more effective way could be to stage the allocation of the good so that each stage reveals information about the participants. We show that a sequence of English auctions, with high reserve prices in early rounds, is a procedure with this property that is more efficient than any one-stage entry auction. Moreover, it also dominates the Anglo-Dutch auction in terms of seller's revenues.

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1 Introduction

One of the salient features of the recent wave of spectrum license auctions has been the disparity of prices across experiences. As an example, the Dutch auction (meaning, the auction in the Netherlands) of licenses for the new UMTS fetched less than a third than the corresponding British or German auctions in per capita terms.[1] One of the explanations proposed for such disparity is the differences across countries with respect to the ratio of incumbents to licenses, and the effects of this ratio on entry and competition for licenses. (See, for instance, Klemperer, 1998b and Milgrom, 2004 .)

Certainly, insufficient entry may limit competition during the bidding process, leading to a low price. The entry decision of a firm depends on a comparison between its costs of participation and its expected benefits. For a potential bidder the costs may include resources needed to assess the value of the license, interest foregone on deposits, etc. These are sunk costs that a firm incurs before it knows whether it wins a license. Therefore, the firm will decide to participate only if it believes that its odds to win the auction are sufficiently high.

But, why is the ratio of incumbents to licenses important in this regard? First, incumbents are strong competitors. Indeed, they may have a client base and lower expected network roll-out cost. That is, their expected valuation of a license is higher. Also, their cost of market prospective may be significantly lower. If there are at least as many incumbents as licenses, then entrants will probably assess as too likely that all licenses will end up in the hands of incumbents, and then entry is not likely to occur. That needs not be either inefficient or problematic in terms of expected license revenue if there are more incumbents than licenses. Yet, if the number of incumbents and licenses coincide, the lack of entry will destroy all sources of competition, and that will have an enormous effect on revenue.

When attracting entrants is a goal, the allocation mechanism should favor entrants over incumbents. For instance, Dutch (or first-price) auctions tilt the allocation in favor of ex-ante weaker bidders, and thus weaker bidders prefer Dutch auctions to efficient, English auctions (see Maskin and Riley, 2000). Based on this fundamental insight, Paul Klemperer (see Klemperer, 1998a) and others have proposed the use of the so-called Anglo-Dutch auction

[1] The Dutch auction took place after the British auction and before the German auction. Many different factors, besides the one pointed out here, may have played a role in producing such marked differences.

when a number of identical objects (licenses) are to be allocated and an identical number of ex-ante stronger incumbents are potential buyers.[2] An Anglo-Dutch auction is a mixture of the two types of auction. It begins with an 'English' phase during which the price rises until all but a number of bidders that exceeds by one the number of objects drop out. At this moment (and price), the auction switches to a second 'Dutch' phase. In this stage, only the remaining bidders can submit (simultaneous, sealed) bids and only bids above the price at which the English phase stopped are allowed.

The first goal of this paper is to show in a very simple model how this auction indeed improves the expected revenues of the seller at the cost of sacrificing efficiency. But our main goal is to investigate other simple alternatives that dominate, both in terms of efficiency and revenues, the Anglo-Dutch auction. We propose what we could term Anglo-Anglo auction: a two-stage, English auction. The design is inspired by Burguet and Sákovics, 1996, and consists of two English phases, the first one run with a (relatively high) reserve price. Instead of using inefficiencies or allocation preference as the tool to induce entry, what the two-stage, English auction uses is the information conveyed by the (absence of) bidding in the first phase. Indeed, if some of the participants in the first phase (incumbents included) are unwilling to bid above the reserve price, they will be perceived as "weaker than expected" bidders. Thus, potential entrants that did not venture to enter in the first round may now consider doing so for the second round.

This is the fundamental insight of this paper. Introducing dynamic elements in the design allows the seller to tilt the information, not the allocation, in favor of potential entrants. As compared to inducing inefficiencies, this may be a more profitable way to overcome the barrier to entry that incumbency represents.

We show that a two-stage, English auction is more efficient than both the English and the Anglo-Dutch auctions. By allowing entry conditional on some private information (entry conditional on bidding behavior), the two-stage entry auction improves upon the most efficient one-stage entry auction, namely, the English auction. Moreover, we show that the gain in efficiency (entry) benefits the seller as well. Indeed, the revenues for the seller are higher in the two-stage, English auction than in the Anglo-Dutch auction.

The analysis is carried out in an extremely simple model, presented in

[2] Paul Klemperer has also emphasized the virtues of Anglo-Dutch auctions when dealing with other issues as well, like collusion, risk aversion, etc.

Section 2, where entrants have to incur a cost before learning their valuations. Moreover, valuations can take only two different values, although incumbents have a higher probability of high valuation. We simplify further by assuming only one unit for sale and one incumbent. The analysis of this model and the results are presented in Section 3. Subsequently, we extend the model in two main directions: multiple units and continuous valuations.

The extension to continuous types is relevant. Indeed, in the two-types model, the Anglo-Dutch is allocationally efficient, in the sense that the object is never assigned to a bidder that competes against a bidder with higher valuation. That is, the only inefficiency may come from inappropriate (excessive) entry. The allocation is still tilted in favor of the ex-ante weaker bidder: when two bidders have the same type, an event with positive probability, the ex-ante weaker bidder will win the auction (will bid higher) with higher probability. Yet one can suspect that by not considering the allocation inefficiency (a tool to reduce informational rents of ex-ante stronger bidders) of Dutch auctions, we underestimate their revenue generation potential. The analysis of the continuous valuations case necessarily relies on numerical computations, since there is no analytical solution to asymmetric Dutch auctions (the second stage of an Anglo-Dutch auction) and the bidding behavior (when to bid) in a two-stage English auction has no simple closed form. Using numerical methods for a family of simple continuous distributions, we obtain the same results as in the base model.

Then we consider multiple units (and the same number of incumbents). Here entry decisions in the second stage of the two-stage, English auction depends on the number of units that are sold in the first stage. The larger the number of units left unsold the larger the number of entrants in the second stage. In fact, under very extreme values of the parameters, the effect that the number of units available has on entry is very extreme. Only in such cases can Anglo-Dutch auctions dominate in terms of revenues the two-stage, English auction. Otherwise, our results for the one unit case hold in the multiple unit case.

2 Rules of the auctions

There are q identical units available for sale and the same number of incumbents. Besides there is a sufficiently large number of potential entrants. In order to learn his valuation and prepare his bid, an entrant has to incur a

cost c. To simplify the analysis, we assume that incumbents already know their valuations and incur no further cost of participation. Each bidder has demand only for one unit. Valuations are private and independently distributed. The valuations of incumbents and entrants are drawn from distribution functions $F_1(v)$ and $F_2(v)$, respectively.

Rules of the Anglo-Dutch auction: Before bidding starts entrants decide whether or not to incur cost c and to learn their valuations.[3] The auction starts as an English auction where bidders continuously raise their bids. We use the clock modelling, so that once a bidder drops he cannot reenter the auction. When $q+1$ bidders are left, the auction switches to the Dutch auction, or more precisely, to the discriminatory auction, which is a generalization of first-price auction for multiple units. Thus, surviving bidders simultaneously bid in this stage, and the q bidders with the highest bids will win one license each. Winners pay their bids. In this Dutch stage, the price at which the last bidder dropped out in the English stage is set as a reserve price or minimum acceptable bid.

In the discrete valuation case, several bidders may drop out from the English auction at a given price, leaving less than $q+1$ bidders active. In that case, we will assume that some of these simultaneously dropping bidders are randomly selected to participate in the Dutch stage so that $q+1$ bidders are still present.

Rules of the two-stage English auction: In the first stage the seller sets a reserve price r, common to all units, and then entrants decide whether to enter or not. Then the oral auction starts. Units are awarded to, at most, the q last bidders to drop. The price is the maximum of r and the price at which only q bidders stay. If less than q bidders offer the reserve price, so that some units fail to sell in the first stage, these remaining units are auctioned in the second stage with the reserve price now set equal to zero. Before this second stage starts, entrants who did not enter in the first stage have a new chance to enter and compete for the remaining units against incumbents and first-stage entrants who abstained from bidding in the first stage. Again, if q' units were left unsold after the first stage, they are awarded to the q' bidders

[3] We will consider pure strategy equilibria in entry. That allows us to leave the total number of potential entrants unspecified. Thus, we will assume at the outset some exogenous order of entry that entrants would consider natural.

at the price at which only these many bidders are left.

3 Discrete valuations, one unit case

Each bidder's valuation may be $\overline{v}$ or $\underline{v}$ with $\overline{v} > \underline{v}$. Bidder 1, the incumbent, has probability μ_1 of having the high valuation $\overline{v}$, and all other bidders (entrants) have probability μ_2 of having high valuation. We assume that $\mu_1 > \mu_2$.

3.1 Anglo-Dutch auction

Assume that, apart from the incumbent, bidder 1, n bidders enter the auction. The English stage will stop at the price at which only two bidders remain in the auction. Also, in this stage bidders' weakly dominant strategy is to stay in the auction until the price reaches their valuations. Thus, if there are more than two bidders with valuation $\overline{v}$, the price will continue increasing until it reaches this value. In such case, the English stage stops at price $\overline{v}$, two bidders are randomly selected to bid in the Dutch part, where they bid $\overline{v}$. On the other hand, if two or less bidders have valuation $\overline{v}$, the English stage stops at price $\underline{v}$. In this case, bidders with high valuation, if any, stay for the Dutch stage. If there are less than two bidders with high valuation, then bidders among dropping (low valuation) bidders are randomly selected so that two bidders take part in the Dutch stage. We assume that bidders recognize each other, so that the identity of bidder 1 known. However, when the English stage stops at price $\underline{v}$, the participants in the Dutch part cannot be sure whether their opponent has been randomly selected among the dropping bidders (i.e., has a low valuation) or not (i.e., has a high valuation).

3.1.1 Bidding in the Dutch stage

We only need to analyze subgames in which the clock stops at price $\underline{v}$ in the English stage. Assume bidder 1 and an entrant play the Dutch stage. It is easy to see that there could be no pure strategy equilibrium in the bidding game. Also, in equilibrium bidders bid $\underline{v}$, the reserve price set by the English stage, if that is their valuation. For valuation $\overline{v}$ bidders, we consider only bidding in an interval. It is easy to rule out bidding (with positive probability) strictly above the supremum of the set of bids that

the rival use with positive probability. That is, this supremum needs being common to both bidders. Also, it is equally easy to rule out mass points at such supremum, as is easy to rule out mass points (or "holes") anywhere in the interior of the intervals of bids used in equilibrium. Finally, equilibrium where the infimum of the intervals are different, or different from $\underline{v}$ can be ruled out as well: there is no point in bidding in the interior of an open interval with zero probability of containing a rival bid. Finally, we can rule out that both bidders' strategies contain a mass point at $\underline{v}$, but we cannot rule out that one of the bidders' strategy has a mass point there.

To avoid open-set problems, we assume that in case two bidders with different valuations tie in the Dutch auction then the winner is the one with higher valuation. We will comment on this tie-breaking rule later.

Thus, let us characterize an equilibrium where bidders bid $\underline{v}$ when their valuation is $\underline{v}$, and bid on $[\underline{v}, \overline{b}]$ when their valuation is $\overline{v}$, for some $\overline{b}$. The entrant bids according to a distribution H_2, and the incumbent, bidder 1, bids according to a distribution H_1. Clearly, bidders with valuation $\underline{v}$ expect zero profits. Since $\overline{b}$ is common and H_i, for $i = 1, 2$, has no atoms at $\overline{b}$ then the expected profits of either an entrant or the incumbent that has valuation $\overline{v}$ when the clock stops at price $\underline{v}$ in the English stage equals $[\overline{v} - \overline{b}]$. Indeed, by bidding $\overline{b}$ either bidder expects to win with probability 1. Since expected profits should be independent on the pure strategy played in a mixed strategy equilibrium, we conclude that the entrant and the incumbent will expect the same equilibrium profit upon entry if they have the same valuation.

This is in fact the way the Anglo-Dutch auction is expected to foster entry. Indeed, notice that entrants with high valuation expect lower profits than incumbents in a standard English auction, since the rival has higher expected valuation. The Dutch stage tilts competition in favor of entrants, and in the case of discrete valuations, this is enough to perfectly level the field.

In order to characterize $\overline{b}$ and H_i, for $i = 1, 2$, we need to consider the posterior beliefs for each type of bidder about the rival bidder. Conditioning of being one of the two bidders in the Dutch auction when the English auction stopped at $\underline{v}$, from the point of view of a rival with high type, an entrant's type is $\overline{v}$ with probability

$$\gamma_2 = \frac{\mu_2}{\mu_2 + \frac{(1-\mu_2)}{n}}.$$

Indeed, given that stopping price (i.e., conditioning on no more than 1 rival

bidder having high valuation), the rival knows that the entrant would be one of the participants in the Dutch stage for sure if his valuation is high, and with probability $\frac{1}{n}$ in case his valuation is (as everybody else's) low. Similarly, the rival with high valuation updates his beliefs about the incumbent having high valuation assigning this event a probability

$$\gamma_1 = \frac{\mu_1}{\mu_1 + \frac{(1-\mu_1)}{n}}.$$

Observe that $\gamma_1 > \gamma_2$. With these posteriors, the expected profit for bidder 1 is, for all b on $(\underline{v}, \overline{b}]$,

$$\pi_1(b) = [(1-\gamma_2) + \gamma_2 H_2(b)]\,(\overline{v} - b). \tag{1}$$

Similarly, for bidder 2, and for all b in $(\underline{v}, \overline{b}]$

$$\pi_2(b) = [(1-\gamma_1) + \gamma_1 H_1(b)]\,(\overline{v} - b). \tag{2}$$

Thus, given $\overline{b}$, (1) and (2) characterize H_i, for $i = 1, 2$. Notice that since $\pi_1(b) = \pi_2(b)$ but $\gamma_1 > \gamma_2$, $H_2(b) < H_1(b)$ for all b. That is, as is usually the case in the Dutch auction, the ex-ante weaker bidder bids more aggressively. Now we apply the condition that the infimum of the intervals of both mixed strategies should be $\underline{v}$. Since only one bidder can have a mass point at that infimum, and $H_2(b) < H_1(b)$ for all b, we conclude that $H_2(\underline{v}) = 0$. That is, from (1)

$$\pi_1 = (1-\gamma_2)\,(\overline{v} - \underline{v}) = \pi_2. \tag{3}$$

Substituting in (1) and (2), we obtain

$$H_2(b) = \frac{1-\gamma_2}{\gamma_2}\frac{b-\underline{v}}{\overline{v}-b}.$$

and

$$H_1(b) = \frac{1-\gamma_1}{\gamma_1}\frac{b-\underline{v}}{\overline{v}-b} + \frac{\gamma_1-\gamma_2}{\gamma_1}\frac{\overline{v}-\underline{v}}{\overline{v}-b},$$

so that

$$H_1(\underline{v}) = \frac{\gamma_1-\gamma_2}{\gamma_1}.$$

Also, using $H_2(\overline{b}) = 1$ we obtain

$$\overline{b} = \gamma_2\overline{v} + (1-\gamma_2)\,\underline{v}.$$

The equilibrium when two entrants play the Dutch auction is even simpler. Here both bidders are symmetric and update according to γ_2 the probability that the rival has high type when he himself does. Then, their expected profits are given by (1), and using $H_2(\underline{v}) = 0$ we obtain that these profits equal (3). This implies that $H_2(b)$ obtained above still describes the equilibrium, except that now this is the common bidding strategy for both rivals.

We can summarize all that has been obtained in the following

Lemma 1 *If selected to play the Dutch stage when the English stage stops at price* $\underline{v}$*,*

i) entrants expect zero profits if their valuation is $\underline{v}$ *and profits* $(1-\gamma_2)(\overline{v}-\underline{v})$ *if their valuation is* $\overline{v}$ *independently of the identity of the rival,*

ii) entrants bid independently of the identity of the rival, but more aggressively than the incumbent.

We should comment now on the role of our tie-breaking assumption. We assume that in case two bidders bid $\underline{v}$, which occurs with positive probability when the incumbent has valuation $\overline{v}$ and the entrant has valuation $\underline{v}$, the bidder with high valuation wins. This allows the latter to bid $\underline{v}$ and still expect to win with probability γ_2. That is, this allows the incumbent to identify the lowest bid that still allows him to defeat the bid of a rival with type $\underline{v}$. Therefore, the tie-breaking assumption comes to play the role of a smallest unit of money. Notice, however, that we do not need assuming that the incumbent has preference when bidding against an entrant with type $\overline{v}$. That is, an entrant that may defeat the incumbent.

We now turn to the analysis of entry and the English stage.

3.1.2 Entry and bidding in the English stage

First we compute the profits that an entrant expects net of the entry cost. Notice that the entrant expects positive profits only when his type is $\overline{v}$ and no more than one other bidder has this type. His profits in this case are independent of the pure strategy (among the ones that belong to the support of his equilibrium mixed strategy) that he chooses to play. One of these strategies is to bid $b = \underline{v}$, in which case he wins $(\overline{v} - \underline{v})$ when all rivals have $\underline{v}$ or when the rival is the incumbent and bids $\underline{v}$. This event has probability

$$\left[(1-\mu_1) + \mu_1 H_1(\underline{v})\right](1-\mu_2)^{n-1}.$$

Thus, substituting for $H_1(\underline{v})$, the expected profits of an entrant are

$$\Pi^a(n) = \mu_2 (1-\mu_2)^{n-1} \left[\frac{(\mu_1 - \mu_2)}{(n-1)\mu_2 + 1} + (1-\mu_1) \right] (\overline{v} - \underline{v}) - c. \qquad (4)$$

This is a decreasing function of n. Entry occurs to the point where the above expression is non negative, and the same expression is negative for $n+1$. That is, treating n as a continuous variable, the number of entrants in the Anglo-Dutch auction, n^a, satisfies $\Pi^a(n^a) = 0$.

Compare this entry decision with the entry decision in a standard English auction. Again, treating n as a continuous variable, the number of entrants in a standard English auction, n^e, solves

$$\mu_2 (1-\mu_1) (1-\mu_2)^{n^e - 1} (\overline{v} - \underline{v}) - c = 0. \qquad (5)$$

Since $\frac{(\mu_1 - \mu_2)}{(n-1)\mu_2 + 1} > 0$, and $\Pi^a(n)$ is decreasing in n, we conclude that

Lemma 2 *The Anglo-Dutch auction promotes entry beyond what is obtained in the standard English auction: $n^a \geq n^e$.*

To conclude with the Anglo-Dutch auction, we can compute the profits expected by the incumbent. Again, planning to bid $\underline{v}$ in the Dutch stage when his type is $\overline{v}$ and the clock stops at $\underline{v}$ at the English stage, the incumbent's expects profits are

$$\mu_1 (1-\mu_2)^n (\overline{v} - \underline{v}). \qquad (6)$$

3.2 Two-stage English auction

Assume that, besides the incumbent, k new entrants enter and learn their valuations in the first stage and, if nobody bids, that is, if all bidders drop before the reserve price r is called, then some additional l bidders enter in the second stage. We look for a separating equilibrium where high valuation bidders of the first stage prefer to bid (i.e., prefer to stay past the reserve price r) while low valuation bidders abstain from bidding in the first stage.[4]

[4]Entrants that would not "bid" in the first stage even if their valuation is high would not enter. Thus, the only pooling equilibrium having no "bids" in the first stage is one where no entrants enter and all wait until the second stage. We will consider this case below.

Thus, assume all bidders (entrants and incumbent) with valuation $\overline{v}$ stay past the reserve price $r \in (\underline{v}, \overline{v})$. Then it is weakly dominant to drop only when the price in the first stage reaches $\overline{v}$, since dropping before implies zero profits. Bidding in the second stage for all $k + l + 1$ participants is simple: again it is weakly dominant to drop at a price equal to valuation.

Apart from entry, the only other important choice for a bidder with valuation $\overline{v}$ present at the first stage is between participating in this first stage and waiting in the hope that there is a second one. The incumbent will prefer to bid in this first stage if

$$(1 - \mu_2)^k(\overline{v} - r) \geq (1 - \mu_2)^{k+l}(\overline{v} - \underline{v}).$$

Indeed, in either case he will earn positive profits, $(\overline{v} - \underline{v})$ in the second stage or $(\overline{v} - r)$ in the first, only if no other participant has valuation $\overline{v}$. Similarly, first stage entrants with high type will prefer to bid if:

$$(1 - \mu_1)(1 - \mu_2)^{k-1}(\overline{v} - r) \geq (1 - \mu_1)(1 - \mu_2)^{k+l-1}(\overline{v} - \underline{v}).$$

In both cases, the restriction can be written as:

$$r \leq \overline{v} - (1 - \mu_2)^l(\overline{v} - \underline{v}). \tag{7}$$

We now turn to the entry decisions in each stage. If the second stage of the auction takes place, potential new entrants learn that the incumbent and the k entrants in the first stage all have low valuations. Thus, treating l as a continuous variable, it satisfies[5]

$$\mu_2(1 - \mu_2)^{l-1}(\overline{v} - \underline{v}) = c. \tag{8}$$

Notice that l is independent of r. Similarly, the zero profit (entry) condition in the first stage is

$$\mu_2(1 - \mu_1)(1 - \mu_2)^{k-1}(\overline{v} - r) = c. \tag{9}$$

Certainly k and l are integers, and thus in general neither of the conditions above are satisfied with equality. That is, entrants expect "some" positive

[5] In fact, it would be given by

$$l = \max\{m | \mu_2(1 - \mu_2)^{m-1}(\overline{v} - \underline{v}) \geq c\}.$$

profits. Again, if there is a "natural" order for potential entrants to enter, as we are assuming, and expected profits in the first stage are no lower than expected profits in the second this creates no additional coordination problems. Also, the seller could consider a small entry in the second stage to keep entrants at their indifference level.

If we compare (8) with (4), we can conclude that $l \geq n^a$. That is, the number of new entrants in the second stage is at least equal to the number of entrants in the Anglo-Dutch auction.

Let us define $r(k)$, for $k = 1, 2...$ as the solution in r of (9) above. Notice that $r(k)$ is decreasing in k. Also, denote by $r^{\times}(0)$ the solution to (7) with equality. This is the highest reserve price compatible with the incumbent bidding in the first stage. Notice that $r^{\times}(0) > r(1)$. Indeed, using (8) we have

$$r^{\times}(0) = \overline{v} - \frac{1-\mu_2}{\mu_2}c,$$

whereas $r(1)$, substituting in (9) for $k = 1$, is

$$r(1) = \overline{v} - \frac{1}{\mu_2(1-\mu_1)}c.$$

Finally, define the reserve price $r(0)$ as

$$r(0) = \overline{v} - \frac{1-\mu_2}{\mu_2(1-\mu_1)}c.$$

It is straightforward to check that at $r(0)$ an incumbent with high valuation that bids in the first stage of the two-stage English auction expects the same profits as in a standard English auction. Notice that $r^{\times}(0) > r(0) > r(1)$.

We should note that for any reserve price in $[r(0), r^{\times}(0)]$ there exist two equilibria. In one of them, the incumbent is expected not to participate in the first stage no matter what valuation he has, and therefore the first stage never sells the license. Thus, the second stage (and the whole two-stage English auction) becomes a regular English auction. (This equilibrium does not exist for reserve prices below $r(0)$, since whatever entrants conjecture in the second stage the incumbent prefers bidding in the first.) In the other equilibrium the incumbent is expected to bid in the first round if his valuation is high, so that in the second stage l new entrants enter. Given this, the incumbent indeed prefers bidding in the first stage when his valuation is high. The first equilibrium does not exist for $r \in (\underline{v}, r(0))$. For the moment we will consider reserve prices in this range. For the case in which the region $[r(0), r^{\times}(0)]$ is important, see the remark at the end of this section.

3.3 Comparing total surplus

One feature of both the Anglo-Dutch auction and the two-stage English auction in this setting is that the license is assigned to the user that values it most among the ones present at the round in which it is assigned. In a more general, continuous type model this is true for the two-stage English auction, but not for the Anglo-Dutch auction. Nevertheless, in our setting efficiency comparisons depend only on the entry decisions.

The standard English auction maximizes the gains from trade among the mechanisms at which entry occurs only at one point in time. Indeed, given n entrants, a new entrant adds surplus only if the $n-1$ previous entrants and the incumbent all have valuation $\underline{v}$ and the new entrant has valuation $\overline{v}$. This event has probability $\mu_2(1-\mu_1)(1-\mu_2)^{n-1}$ and the increase in surplus is $(\overline{v}-\underline{v})$ in this case. Trading this increased expected surplus with the cost of entry c results in (5), the entry condition in an English auction. In this sense, the Anglo-Dutch auction fosters entry beyond what is efficient.

In a two-stage English auction, entry may take place at more than one point in time. If the license is not assigned in the first stage, then new entrants will enter to take part in a final, English auction. This second-stage entry conditional on all first-stage participants having low type $\underline{v}$ is also (conditionally) efficient. Indeed, the expected surplus, given that there is a second stage, is

$$\underline{v} + \left[1-(1-\mu_2)^l\right](\overline{v}-\underline{v}) - lc$$

where l is given by (8). Now,

$$\left[1-(1-\mu_2)^l\right](\overline{v}-\underline{v}) = (\overline{v}-\underline{v})\sum_{m=1}^{l}\mu_2(1-\mu_2)^{m-1}.$$

Notice that there are l terms on the right hand side, and they are decreasing in m. Thus, the smallest one is $\mu_2(1-\mu_2)^{l-1}(\overline{v}-\underline{v})$. But this term is equal to c, if (8) is satisfied.

It should not come as a surprise that a two-stage entry mechanism may result in a higher surplus than even the most efficient one-shot entry. In fact, this is so for any reserve price choices.

Proposition 3 *The expected surplus in any two-stage English auction is higher than in a standard English auction and therefore also higher than in an Anglo-Dutch auction.*

Proof. We have already established that the second stage entry l is conditionally efficient, i.e., that l minimizes $(1-\mu_2)^l(\overline{v}-\underline{v})+lc$, also that $l \geq n^e$, and that l is independent of the reserve price, and therefore independent of first stage entry k. Notice that for $r = \underline{v}+\epsilon$ and ϵ small, (9) is (virtually) the entry condition in the English auction, therefore $k \leq n^e$ for any r. Now, the total net surplus from a two-stage English auction given entry decisions $k \leq n^e$ is

$$\begin{aligned}
& \overline{v} - (1-\mu_1)(1-\mu_2)^k\left[(1-\mu_2)^l(\overline{v}-\underline{v})+lc\right] - kc \\
> \; & \overline{v} - (1-\mu_1)(1-\mu_2)^k\left[(1-\mu_2)^{n^e-k}(\overline{v}-\underline{v})+(n^e-k)c\right] - kc \\
= \; & \overline{v} - (1-\mu_1)(1-\mu_2)^{n^e}(\overline{v}-\underline{v}) - (1-\mu_1)(1-\mu_2)^k(n^e-k)c - kc \\
> \; & \overline{v} - (1-\mu_1)(1-\mu_2)^{n^e}(\overline{v}-\underline{v}) - n^e c,
\end{aligned}$$

where the last line is the expected surplus in the standard English auction. QED ■

According to the above proposition, any reserve price, including $r = \underline{v}$ $(+\epsilon)$, the one that maximizes entry in the first round and still allows a positive probability of new entry in the second round, induces more efficient entry than the most efficient one-shot entry auction. But what is the efficient level of first-stage entry in the two-stage English auction? The answer to this question will also be relevant when discussing revenues. When there is a second opportunity to experiment, i.e., to obtain valuation draws, assigning the license in the first round has an opportunity cost above $\underline{v}$. Then efficient entry in the first stage needs not be the highest compatible with screening low valuation types. Indeed,

Lemma 4 *Maximizing surplus in a two-stage English auction requires limiting entry. In particular, efficient entry k^* satisfies*

$$r(k^*) = \underline{v} + (\overline{v}-\underline{v})\left[1-(1-\mu_2)^{l-1}(1+(l-1)\mu_2)\right]. \tag{10}$$

Proof. The marginal contribution of a new entrant in the first stage of the two stage English auction is

$$\mu_2(1-\mu_1)(1-\mu_2)^{k-1}[(1-\mu_2)^l(\overline{v}-\underline{v})+lc]-c. \tag{11}$$

Indeed, $\mu_2(1-\mu_1)(1-\mu_2)^{k-1}$ is the probability that the new entrant has a high valuation $\overline{v}$, and the rest of entrants have low valuation $\underline{v}$. In this case,

we would have higher (gross) surplus with the additional entrant in the first stage if future entrants were to have low type as well, which has probability $(1-\mu_2)^l$. The additional (gross) surplus would be $(\overline{v}-\underline{v})$, but entry of the second period entrants would also imply a cost lc, which a good realization of a first stage entrant would save. Now, treating k and l as continuous variables and substituting equation (8), entry in the first stage should take place until the point where

$$\mu_2(1-\mu_1)(1-\mu_2)^{k-1}(1-\mu_2)^{l-1}(\overline{v}-\underline{v})(1+(l-1)\mu_2)=c.$$

It follows that efficient first stage entry $k^* < n^e$**.** Now, $\mu_2(1-\mu_1)(1-\mu_2)^{k-1}(\overline{v}-r)=c$. Substituting in (9), we obtain (10).QED ■

3.4 Comparing revenues

Revenues and efficiency are intimately related. Indeed, disregarding the integer problem, entrants expect zero profits (net of the entry cost) both in a standard English, an Anglo-Dutch, and a two-stage English auctions. Therefore, we need only looking at total surplus (net of entry costs) and the profits of the incumbent when comparing the seller's revenues in both auctions. The incumbent's profits are $\mu_1(1-\mu_2)^n(\overline{v}-\underline{v})$ both in the standard English and in the Anglo-Dutch auctions, except that n may differ in both. Thus, the revenues for the seller in each case are

$$R(n^i)=\overline{v}-(1-\mu_2)^{n^i}(\overline{v}-\underline{v})-n^ic,$$

for $i=e,a$. We can compute $R(n)-R(n-1)$ for any n to obtain

$$R(n)-R(n-1)=\mu_2(1-\mu_2)^{n-1}(\overline{v}-\underline{v})-c.$$

This is decreasing in n. Thus, $R(n)$ is maximized when this expression is zero, i.e., when $n=l$ (again, treating n as a continuous variable), increasing for $n<l$, and decreasing for $n>l$. Compare this with entry decisions in Anglo-Dutch and English auctions, obtained respectively from (4) and (5) above. Since $l>n^a>n^e$,

Proposition 5 *The revenues of the seller are higher in the Anglo-Dutch auction than in the standard English auction.*

As conjectured, the Anglo-Dutch auction increases the revenues of the seller by increasing entry.

Consider now the two-stage English auction. When maximizing revenues, the seller needs only consider the maximum of the reserve prices compatible with any amount of entry, k, that is, $r(k)$ defined above. Indeed, any two values for the reserve price that induce the same first period entry (and also the same second period entry) also induce the same total (gross) surplus and the same cost of entry, whereas both the profits of first period entrants and incumbents are lower for the highest of the two reserve prices.

Then, let us compute the expected profit of the incumbent, i.e., $\mu_1(1-\mu_2)^k(\overline{v}-r(k))$ for different values of k. Substituting (9), we can write this expected profits as

$$\frac{\mu_1}{1-\mu_1}\frac{1-\mu_2}{\mu_2}c.$$

Thus,

Lemma 6 *The expected profits of the incumbent evaluated at $r(k)$, are independent of k.*

In other words, from the point of view of the seller's revenues, the reduction in reserve price that is necessary to attract one more entrant in the first round exactly compensates the increase of competition obtained in this way, starting from any level of entry $k \geq 1$. Thus, we have

Corollary 7 *From the point of view of the seller, the optimal reserve price is $r(k^*)$, which also maximizes total surplus.*

Remark: If we select the equilibrium where the incumbent with high valuation bids in the first stage, in the range $[r(0), r^{\times}(0)]$, then $r^{\times}(0)$ may result in higher revenues for the seller. [6]

[6] Notice that this amounts to make a take it or leave it offer to the incumbent. If the seller has the ability to exclude bidders from future stages, this may be optimal. In fact, McAfee and McMillan, 1988 have shown in a model that could be reduced to our model except for its symmetry, that the optimal mechanism for a seller with this ability when the number of potential buyers is unbounded is to induce one by one entry and offer each entrant a (constant accross periods) price. With a finite number of potential entrants, this reserve price would have to be decreasing (see Burguet, 1996), so that buyers could buy in future periods even if they reject the offer at the time they enter. The mechanism, however, would have to be complemented with asymmetric subsidies even when buyers are symmetric.

We are now ready to compare seller's revenues in the Anglo-Dutch auction and in the two-stage English auction.

Proposition 8 *A two-stage English auction with appropriate reserve price $r(k^*)$ results in higher revenues for the seller than the Anglo-Dutch auction.*

Proof. The revenues for the seller in an Anglo-Dutch auction are

$$R^{AD} = R(n^a) = \overline{v} - (1-\mu_2)^{n^a}(\overline{v} - \underline{v}) - n^a c,$$

whereas the revenues of the seller in a two-stage English auction with $r = r(k^*)$, using the definition of $r(k^*)$, can be written as

$$\begin{aligned} R^{2S}(r(k^*)) &= \overline{v} - (1-\mu_1)(1-\mu_2)^{k^*+l}(\overline{v} - \underline{v}) - k^* c - \\ &\quad (1-\mu_1)(1-\mu_2)^{k^*} l c - \frac{\mu_1}{1-\mu_1}\frac{1-\mu_2}{\mu_2} c, \end{aligned}$$

where the last term represents the profits of the incumbent. Then, since at k^*, (11) equals zero, substituting this expression for c in that last term, we have

$$R^{2S}(r(k^*)) = \overline{v} - (1-\mu_2)^{k^*}\left[(1-\mu_2)^l(\overline{v} - \underline{v}) + lc\right] - k^* c.$$

Notice that, for all $m \leq l$,

$$\begin{aligned} (1-\mu_2)^m(\overline{v} - \underline{v}) + mc &= \left[(1-\mu_2)^{m-1} - \mu_2(1-\mu_2)^{m-1}\right](\overline{v} - \underline{v}) + mc \leq \\ &\quad (1-\mu_2)^{m-1}(\overline{v} - \underline{v}) + (m-1)c. \end{aligned}$$

Thus, repeating this for $m = l, l-1, ..., n^a - k^*$,

$$\begin{aligned} R^{2S}(r(k^*)) &\geq \overline{v} - (1-\mu_2)^{k^*}\left[(1-\mu_2)^{n^a - k^*}(\overline{v} - \underline{v}) + (n^a - k^*)c\right] - k^* c \\ &= \overline{v} - (1-\mu_2)^{n^a}(\overline{v} - \underline{v}) - \left[(1-\mu_2)^{k^*}(n^a - k^*) + k^*\right] c \geq R^{AD}. \end{aligned}$$

QED ■

In an Anglo-Dutch auction, the seller sacrifices surplus to foster entry and obtain higher revenues. A two-stage English auction increases the revenues of the seller by improving the efficiency of entry decisions. As a result, both the revenues of the seller and the efficiency of the allocation are higher than what they are in a Anglo-Dutch auction.

Remark: We have been assuming (implicitly, by treating entry variables as continuous) that at least for $k = 0$, $r(k) > \underline{v}$. It may well happen that

this is not the case. That is, it may happen that $(\overline{v}-\underline{v})\mu_2(1-\mu_1) < c$.[7] It may also happen that entry is positive in the Anglo-Dutch auction, i.e., $(\overline{v}-\underline{v})\mu_2(1-\mu_2) \geq c$. Notice that in the latter case $r^{\times}(0) > \underline{v}$. That is, for a reserve price above $\underline{v}$ there is an equilibrium where the incumbent is expected to bid in case her valuation is high, although there is also an equilibrium in which the incumbent is not expected to bid. Notice that if $r = r^{\times}(0)$ and the "incumbent-bids" equilibrium is played, then

$$\begin{aligned} R^{2S}(r^{\times}(0)) &= \overline{v} - (\overline{v}-\underline{v})(1-\mu_2)^l - (1-\mu_1)lc \\ &= \overline{v} - (\overline{v}-\underline{v})(1-\mu_2)^l - lc + \mu_1 lc \\ &\geq \overline{v} - (\overline{v}-\underline{v})(1-\mu_2)^{n^a} - n^a c + \mu_1 lc \\ &> \overline{v} - (\overline{v}-\underline{v})(1-\mu_2)^{n^a} - n^a c = R^{AD}, \end{aligned}$$

where the first inequality above follows from the fact that l maximizes $\overline{v} - (\overline{v}-\underline{v})(1-\mu_2)^l - lc$, as we showed in the previous subsection.

Notice that in this simple example, there is an equally simple way to "elect" the "incumbent-bids" equilibrium: the seller can make a take it or leave it offer to the incumbent, r, and then run an auction only with entrants if the incumbent does not accept. In that case, entry in the auction round is equal to l. For any $r \in [\underline{v}, r^{\times}(0)]$, this is equivalent to running the original two-stage English auction with the "incumbent-bids" equilibrium. Certainly, the details of the model, and its simplicity, are key for this result. But in general there will always be a way to "handicap" the incumbent so that even with no entry in the first round information is transmitted to potential entrants so that entry is induced in the second round. In this sense, even though we view the two-stage English auction as an alternative to the Anglo-Dutch auction, certainly introducing certain inefficiencies and managing information leakage may be "complements" rather than "substitutes".

4 Remarks on generalizations

In the previous sections we have analyzed a very stylized model of competition, where buyers' valuations could take one of two specific values. This

[7] Notice that this would always be the case for high enough values of μ_1, close to 1. That is, whenever it is common knowledge that with almost certainty the incumbent has a (weakly) higher valuation than entrants.

was enough to illustrate the main insights behind the proposal to use Anglo-Dutch auctions to foster entry in the presence of a strong incumbent. Indeed, the incumbent bids less aggressively in the Dutch part, so that the probability that an entrant obtains the good (license) at a profit is enhanced. This fosters entry and enhances the revenues for the seller at an efficiency cost: excessive entry. The stylized model also shows that a two-stage English auction may be more appropriate to attain the goal of high revenues with no cost of (and even enhancing of) efficiency.

Yet, there is one aspect of Anglo-Dutch auctions that the discrete case does not reflect: the Dutch stage may introduce inefficiency that goes beyond excessive entry, i.e., allocation inefficiency. Indeed, in asymmetric settings, a Dutch auction may assign the good or license to a buyer different from the one that has the highest willingness to pay. This inefficiency is in general to the advantage of both the entrant and the seller (see Maskin and Riley, 2000).

4.1 Continuous distributions

Thus, the first generalization we may consider is to assume continuous distributions of types, where this allocation inefficiency appears. We assume now that v_i, the valuation of a buyer, is a (independent) realization of a continuous random variable. The incumbent draws his type from a distribution $F_1(v)$, whereas all entrants draw their types from distribution $F_2(v)$. We further assume that F_1 stochastically dominates F_2.

In the Anglo-Dutch auction, it is still weakly dominant for bidders to stay in the auction up to the moment when the price reaches their respective valuations. If the clock stops at price ρ, with two entrants as the remaining bidders, then these bidders participate in a symmetric Dutch auction with reserve price ρ, the one with highest valuation will win, and the revenues for the seller will be equal to the expected value of the second largest valuation. When one of the two remaining bidders is the incumbent, however, bidders participate in an asymmetric Dutch auction. In general, bidding strategies in this case, and expected revenues for the seller, can only be obtained through numerical methods.

With respect to the two-stage English auction, and for any given reserve price set by the seller, r, both the incumbent's and entrants' optimal behavior is to drop at a price equal to their willingness to pay, if they do participate in any of the stages. Thus, we only need analyzing participation decisions. We can conjecture that these will be characterized by two cut-off values, w_1, w_2,

such that the incumbent decides to participate in the first period if $v_1 \geq w_1$, and any first period entrant i participates if $v_i \geq w_2$. Both of these values will depend on the number of entrants in the first period and how many are expected in the second. Treating entry as a continuous variable, then zero profits for entrants in both stages and indifference between participating and waiting both for entrants and the incumbent at their respective cut-off valuations are the four equations that the solution (w_1, w_2, k, l) solves.

Using numerical computations, we have solved for the continuous model with asymmetric, uniform distributions, $F_i(v) = \frac{v}{\overline{v}_i}$, $i = 1, 2$, with $1 = \overline{v}_2 \leq \overline{v}_1$. The details are shown in the Appendix. In all cases analyzed, the total surplus is higher under a two-stage English auction than under an Anglo-Dutch auction. Revenues also follow this ranking except for one case: when $\overline{v}_1 = 1.2$, and $c = 0.072$. This cost was chosen (as in all examples) so that entrants in the Anglo-Dutch auction break even, the most favorable case from the point of view of the seller. In this particular case, 2 entrants enter the Anglo-Dutch auction. For these same values, the optimal reserve price in a two-stage English auction is $r = \frac{2}{3}$, which generates zero entry in the first stage of the two-stage English auction, and 2 entrants in the second stage. ($r = \frac{2}{3}$ is the maximum reserve price compatible with the incumbent "bidding" in the first stage with positive probability, since the expected highest bid from the two second period entrants is $\frac{2}{3}$. Consequently, $w_1 = 1$.) This obviously results in lower revenues for the seller (0.528, instead of 0.531). Why is entry not higher in the two-stage English auction? The answer has to do with the integer nature of entry. Indeed, with $l = 2$, entrants expect positive profits, equal to 0.0116. (Recall that entrants expect zero profits in the Anglo-Dutch auction with these values of the parameters.) Yet with $l = 3$ they would expect negative profits. The seller needs only setting an entry fee in the second stage of the two-stage English auction above 0.0016 (much lower than 0.0116), which entrants would be willing to pay, to improve upon the Anglo-Dutch auction.

4.2 More than one unit

We could also ask whether our results extend to the case where the seller has more than one unit for sale, and the equality between the number of units and the number of incumbents still holds. In the Appendix we consider the case where 2 units are to be sold and there are 2 incumbents. Other things are as in the model analyzed above. The main novelty here is that, in the

second stage of a two-stage English auction, there is now a chance that one of the units for sale is assigned in the first stage, but the other unit is still available in the second stage. Let l_1 be the number of second-stage entrants in case 1 unit is left and l_2 the number of second-stage entrants when 2 units are still available. These are defined by

$$\mu_2(1-\mu_2)^{l_1-1}(\overline{v}-\underline{v}) = c, \tag{12}$$

and

$$\mu_2[(1-\mu_2)^{l_2-1} + (l_2-1)\mu_2(1-\mu_2)^{l_2-2}](\overline{v}-\underline{v}) = c. \tag{13}$$

Notice that the second stage is independent of the identity of the winner of the first, given our assumptions. In the Appendix we show that the socially efficient entry and allocation in one-stage auctions is achieved using, for instance, an English auction. However, a two-stage English auction always results in higher surplus than any one-stage auction. Thus, our results on efficiency carry over to this multiple-unit case.

With respect to the seller's revenues a sufficient condition for the ranking to be the same is $l_2 \leq 2(l_1+1)$. In fact, this leaves a lot of slack. We have obtained cases where the Anglo-Dutch auction performs better than the two-stage English auction. However, these cases involve both extremely low values of μ_2 and c so that even though no entry would take place in an English auction, a large number of firms would enter in an Anglo-Dutch or a second stage of the two-stage English auction.

5 Concluding remarks

We have offered theoretical support to the claim that an Anglo-Dutch auction results in higher revenues than ascending auctions when there are as many licenses as incumbents. By favoring ex-ante weaker entrants, the Anglo-Dutch auction fosters entry and this results in higher prices of licenses at the cost of efficiency. However, we have also proposed another simple alternative to this Anglo-Dutch auction: a two-stage English auction. What we could term an Anglo-Anglo auction. Instead of relying on inefficient allocations to induce entry, the two-stage English auction relies on information revealed through bidding, an information on which entrants can condition their entry decisions. As a result, entry is more efficient and surplus is higher, which

works to the advantage of the seller. Thus, this simple design not only increases revenues but also the gains from trade.

There is one aspect in which the two-stage English auction is more complex than the Anglo-Dutch auction: its intrinsic multistage nature. Indeed, the Anglo-Dutch auction requires two stages, but no lag of time is needed between them. On the contrary, in a two-stage English auction potential new entrants should be given enough time to "enter" (find about their valuation, prepare a bidding strategy) before the second stage takes place. In cases where this waiting is costly, this waiting cost would have to be weighed against the gains that we have discussed. Also, there is another dimension in which the Anglo-Dutch auction is simpler, and then more robust: the seller needs not "computing" reserve prices. As such, the two-stage English auction is less "distribution-free". On the other hand, the two-stage English auction is simpler from the point of view of buyers: under private values, the bidding strategies are straightforward, whereas in Anglo-Dutch auctions they require information about the valuation distributions of opponents. Certainly some information about these distributions is still required when deciding whether to enter a two-stage English auctions, but this is also required for entering an Anglo-Dutch auction. In any case, the conclusion of this work is that the learning that dynamic designs allow may be a better alternative than inefficiencies when the goal is fostering competition through entry.

6 Appendix

6.1 Continuous valuations

Consider the Anglo-Dutch auction. There are initially n entrants plus the incumbent who participate in the English stage. A weakly dominant strategy for bidders is to stay in the auction up to the moment when the price reaches their respective valuations and then to drop out. Suppose that the bidder with third highest valuation drops out at the valuation ρ. Then the remaining two bidders participate in the Dutch stage with reserve price ρ. Suppose that two bidders with the highest valuations are entrants. Then the expected revenue for seller from the Dutch stage is

$$\int_{\rho}^{\overline{v}_2} v(1 - F_2(v)) f_2(v) dv,$$

which is the expected value of the second highest valuation given that it exceeds ρ. The density (on $[0,\rho)$) of the highest valuation ρ (reserve price) among the remaining $n-1$ bidders is

$$f_{n-1:n-1}(\rho) = F_2(\rho)^{n-2} f_1(\rho) + (n-2) F_2(\rho)^{n-3} F_1(\rho) f_2(\rho).$$

There are $n(n-1)$ permutations when two bidders with the highest valuations are entrants. Thus, the revenue for seller (times the probability of the event) accruing when two entrants play the Dutch part is

$$\begin{aligned} R_{w,w} &= n(n-1) \int_0^{\overline{v}_2} \left(\int_\rho^{\overline{v}_2} v(1-F_2(v)) f_2(v) dv \right) dF_2(\rho)^{n-2} F_1(\rho) \quad (14) \\ &= n(n-1) \int_0^{\overline{v}_2} v(1-F_2(v)) F_2(v)^{n-2} F_1(v) f_2(v) dv. \end{aligned}$$

Suppose now that one of the two bidders with the highest valuations is the incumbent. Define the truncated distributions

$$\begin{aligned} G_2(v) &\equiv \frac{F_2(v) - F_2(\rho)}{1 - F_2(\rho)} \\ G_1(v) &\equiv \frac{F_1(v) - F_1(\rho)}{1 - F_1(\rho)} \end{aligned}$$

Let $v = \phi_2(b)$ and $v = \phi_1(b)$ be inverse bid functions, respectively, for entrant and incumbent, defined on $[\rho, b^*]$, such that $\phi_i(\rho) = \rho$ and $\phi_i(b^*) = \overline{v}_i$ for $i = 1, 2$. When an entrant bids b he wins with probability $G_1(\phi_1(b))$ and pays his bid. Similarly, the distribution of an entrant's bid is $G_2(\phi_2(b))$. Therefore the expected revenue for the seller when accruing from an entrant-winner is

$$R_2 = \int_\rho^{b^*} b G_1(\phi_1(b)) dG_2(\phi_2(b)),$$

and the expected revenue accruing from an incumbent-winner is

$$R_1 = \int_\rho^{b^*} b G_2(\phi_2(b)) dG_1(\phi_1(b)).$$

The distribution of ρ is

$$f_{n-1:n-1}(\rho) = (n-1) F_2(\rho)^{n-2} f_2(\rho).$$

Then, the expected revenue accruing to the seller from an entrant when the incumbent is one of the two bidders with the highest valuations is

$$R_{w,s} = n \int_0^{\overline{v}_2} R_2(1 - F_2(q))(1 - F_1(q))(n-1)F_2(q)^{n-2} f_2(q) dq, \qquad (15)$$

and the expected revenue accruing to the seller from the incumbent is

$$R_{s,w} = n \int_0^{\overline{v}_2} R_1(1 - F_2(\rho))(1 - F_1(\rho))(n-1)F_2(\rho)^{n-2} f_2(\rho) d\rho. \qquad (16)$$

Thus, the total revenues of the seller are $R_{w,w} + R_{w,s} + R_{s,w}$. Notice that R_1 and R_2 depend on the (inverse) bidding functions $\phi_i(b)$. These have to be computed using numerical methods.

Now we turn to a two-stage English auction. Again assume that, besides the incumbent, k entrants enter in the first stage and, if nobody bids, some additional l bidders enter in the second stage. Since first stage entrants must bid at least the reserve price r, they will decide to participate in the first stage bidding if their valuations will exceed a cut-off value w_2. Similarly, the incumbent will decide to participate in the bidding if his valuation is above a cut-off value w_1.

Define truncated distributions for $i = 1, 2$ as

$$H_i(v) \equiv \frac{F_i(v)}{F_i(w_i)}.$$

We can distinguish three cases: (1) $\overline{v}_2 \geq w_2 \geq w_1 \geq 0$, (2) $\overline{v}_2 \geq w_1 \geq w_2 \geq 0$, (3) $\overline{v}_1 \geq w_1 \geq \overline{v}_2 \geq w_2 \geq 0$. (And two more separate cases when $k = 0$ since then w_2 does not exist: $\overline{v}_2 \geq w_1 \geq 0$ and $\overline{v}_1 \geq w_1 \geq \overline{v}_2$.) Here we present only derivations of cut-off points for the first case. Note that once bidders decide to bid (both in the first and second stages) it is weakly dominant for them to bid their true valuations. The cut-off point w_1 for the incumbent is found when he is indifferent between obtaining the object in stage 1 at reserve price r or waiting till stage 2 and obtaining it at the highest valuation among $k + l$ entrants. Thus,

$$(w_1 - r)F_2(w_2)^k = \int_0^{w_1} (w_1 - v) dF_2(v)^l F_2(v)^k = \int_0^{w_1} F_2(v)^l F_2(v)^k dv, \quad (17)$$

and the cut-off point w_2 for each of the k entrants satisfies

$$\begin{aligned}&(w_2-r)F_2(w_2)^{k-1}F_1(w_1)+\int_{w_1}^{w_2}(w_2-v)F_2(w_2)^{k-1}dF_1(v) \qquad (18)\\ =\ &\int_{w_1}^{w_2}(w_2-v)F_1(w_1)dF_2(v)^lF_2(v)^{k-1}+\int_0^{w_1}(w_2-v)F_1(v)dF_2(v)^lF_2(v)^{k-1}.\end{aligned}$$

Here we have an extra term since an entrant with valuation w_2 will win against an incumbent whose valuation takes value in (w_1, w_2). Rearranging,

$$\begin{aligned}&(w_1-r)F_2(w_2)^{k-1}F_1(w_1)+\int_{w_1}^{w_2}F_2(w_2)^{k-1}F_1(v)dv\\ =\ &\int_0^{w_1}F_2(v)^lF_2(v)^{k-1}F_1(v)dv+\int_{w_1}^{w_2}F_2(v)^lF_2(v)^{k-1}F_1(w_1)dv.\end{aligned}$$

We can express both conditions using truncated distributions

$$w_1-r=\int_0^{w_1}F_2(v)^lH_2(v)^kdv$$

$$\begin{aligned}&w_1-r+F_1(w_1)^{-1}\int_{w_1}^{w_2}F_1(v)dv\\ =\ &\int_0^{w_1}F_2(v)^lH_2(v)^{k-1}H_1(v)dv+\int_{w_1}^{w_2}F_2(v)^lH_2(v)^{k-1}dv.\end{aligned}$$

Combining both equations gives

$$\begin{aligned}&\int_0^{w_1}F_2(v)^lH_2(v)^kdv+F_1(w_1)^{-1}\int_{w_1}^{w_2}F_1(v)dv \qquad (19)\\ =\ &\int_0^{w_1}F_2(v)^lH_2(v)^{k-1}H_1(v)dv+\int_{w_1}^{w_2}F_2(v)^lH_2(v)^{k-1}dv.\end{aligned}$$

Let us define the revenue of the seller from the incumbent by R_s, from each of first-stage entrants by R_k, and from each of second-stage entrants by R_l. The total revenue to the seller then is $R_s + kR_k + lR_lF_2(w_2)^kF_1(w_1)$. Also define the expected profit of each of first-stage entrants by P_k, and of each of second-stage entrants by P_l.

For fixed entry (k, l) we solve the following maximization problem:

$$\max_{w_1,w_2} R_s(w_1,w_1)+kR_k(w_1,w_2)+lR_l(w_1,w_2)F_1(w_1)F_2(w_2)^k \qquad (20)$$

subject to constraint (19), and inequalities $\overline{v}_2 \geq w_2 \geq w_1 \geq 0$, $P_k(w_2, w_1) \geq c$ and $P_l(w_2, w_1) \geq c$. After solving for this, we can find the reserve price r from either equation (17) or (18). Next we present expressions for revenues R_s, R_k, R_l, and profits P_k, P_l.

The expected revenue from the incumbent is

$$\begin{aligned} R_s &= \int_{\overline{v}_2}^{\overline{v}_1} J_1(v) f_1(v) dv + \int_{w_2}^{\overline{v}_2} J_1(v) F_2(v)^k f_1(v) dv + \\ &\quad \int_{w_1}^{w_2} J_1(v) F_2(w_2)^k f_1(v) dv + \int_0^{w_1} J_1(v) F_2(v)^{k+l} f_1(v) dv, \end{aligned} \tag{21}$$

where, as usual,

$$J_i(v) \equiv v - \frac{1 - F_i(v)}{f_i(v)},$$

for $i = 1, 2$. The expected revenue from each of the k first stage entrants is

$$\begin{aligned} R_k &= \int_{w_2}^{\overline{v}_2} J_2(v) F_1(v) F_2(v)^{k-1} f_2(v) dv + \int_{w_1}^{w_2} J_2(v) F_1(w_1) F_2(v)^{k+l-1} f_2(v) dv + \\ &\quad \int_0^{w_1} J_2(v) F_1(v) F_2(v)^{k+l-1} f_2(v) dv, \end{aligned} \tag{22}$$

and the expected revenue from each of the l second stage entrants is

$$\begin{aligned} R_l &= \int_{w_2}^{\overline{v}_2} J_2(v) f_2(v) dv + \int_{w_1}^{w_2} J_2(v) H_2(v)^k F_2(v)^{l-1} f_2(v) dv + \\ &\quad \int_0^{w_1} J_2(v) H_1(v) H_2(v)^k F_2(v)^{l-1} f_2(v) dv. \end{aligned} \tag{23}$$

The expected profit of each of the k first stage entrants is

$$\begin{aligned} P_k &= \int_{w_2}^{\overline{v}_2} (1 - F_2(v)) F_1(v) F_2(v)^{k-1} dv + \int_{w_1}^{w_2} (1 - F_2(v)) F_1(w_1) F_2(v)^{k+l-1} dv + \\ &\quad \int_0^{w_1} (1 - F_2(v)) F_1(v) F_2(v)^{k+l-1} dv \geq c, \end{aligned} \tag{24}$$

and the expected profit of each of the l second stage entrants is

$$\begin{aligned} P_l &= \int_{w_2}^{\overline{v}_2} (1 - F_2(v)) F_2(v)^{l-1} dv + \int_{w_1}^{w_2} (1 - F_2(v)) H_2(v)^k F_2(v)^{l-1} dv + \\ &\quad \int_0^{w_1} (1 - F_2(v)) H_1(v) H_2(v)^k F_2(v)^{l-1} dv \geq c. \end{aligned} \tag{25}$$

For numerical simulations we assume that valuations come from the uniform distributions on $[0, \overline{v}_i]$

$$F_i(v) = \frac{v}{\overline{v}_i}, \tag{26}$$

with $\overline{v}_2 \leq \overline{v}_1$. The results from numerical simulations are summarized in Table 1. We have fixed $\overline{v}_2 = 1$ for all simulations. Table 1 illustrates results when $\overline{v}_1$ varies. The entry cost c was chosen to ensure that n entrants in Anglo-Dutch auction earn exactly zero net profits. With the uniform distributions, and when first-stage entry is positive, one can show that $w_1 = w_2$ satisfies the equations for cut-off points. Among the results presented in the Table 1 only in one case the revenues of seller are lower in two-stage English auction than in Anglo-Dutch auction, namely, in the auction that does not induce strictly larger (overall) number of entrants than Anglo-Dutch auction: $\overline{v}_1 = 1.2$, $c = 0.072$, where R^{AD} is equal to 0.531, and $R^{2S} = 0.528$. Yet, this is due to an integer problem. Indeed, two entrants expect substantial positive profits in the second stage of the two-stage English auction (0.0116 net of entry cost, in this case), yet a third one would expect negative profits. If we keep the profits of entrants in the second stage to zero, for instance, charging these entrants an entry fee of 0.0116, which would be paid in case the incumbent did not bid (an event with probability 1/1.2), then $R^{2S} = 0.547$ and then the revenues for the seller would again be larger in the two-stage auction.

6.2 The two unit case

Again we show that the two-stage English auction is more efficient than both the standard English auction and the Anglo-Dutch auction for all reserve prices r. The expected surplus for any one-stage auction is given by

$$S^1 = 2\overline{v} - \{2\mu_1(1-\mu_1)(1-\mu_2)^n + (1-\mu_1)^2[2(1-\mu_2)^n + n\mu_2(1-\mu_2)^{n-1}]\}(\overline{v}-\underline{v}) - nc,$$

which is maximized by n such that

$$\mu_2[(1-\mu_1)^2\{(1-\mu_2)^{n-1} + (n-1)\mu_2(1-\mu_2)^{n-2}\} + 2\mu_1(1-\mu_1)(1-\mu_2)^{n-1}](\overline{v}-\underline{v}) = c,$$

and is achieved using an English auction. Thus, necessarily expected surplus from Anglo-Dutch auction is at most equal to the expected surplus from the

Table 1: Numerical simulations of auctions for uniform distributions with $\bar{v}_2 = 1$

$\overline{v}_1$	c	n	S^{AD}	R^{AD}	k	l	w_1	r	S^{2S}	R^{2S}
1	0.083	2	0.583	0.500	1	2	0.552	0.510	0.630	0.529
1	0.050	3	0.650	0.600	1	3	0.700	0.652	0.692	0.631
1	0.024	5	0.738	0.715	2	5	0.771	0.745	0.770	0.745
1.2	0.072	2	0.663	0.531	0	2	1.000	0.667	0.689	0.528
1.2	0.043	3	0.720	0.628	2	4	0.555	0.547	0.754	0.647
1.2	0.020	5	0.795	0.737	3	5	0.711	0.697	0.826	0.751
1.5	0.059	2	0.791	0.570	0	3	0.836	0.714	0.850	0.635
1.5	0.035	3	0.839	0.663	0	4	0.954	0.796	0.883	0.710
1.5	0.016	5	0.901	0.766	0	6	1.000	0.857	0.934	0.786
2	0.045	2	1.018	0.620	0	3	1.000	0.750	1.082	0.675
2	0.027	3	1.057	0.708	0	4	1.000	0.800	1.113	0.733
2	0.012	5	1.107	0.802	0	7	1.000	0.875	1.151	0.826

standard English auction. The expected surplus for the two-stage English auction is given by

$$\begin{aligned} S^2 &= 2\overline{v} - \{2\mu_1(1-\mu_1)(1-\mu_2)^k + (1-\mu_1)^2 k\mu_2(1-\mu_2)^{k-1}\}[(1-\mu_2)^{l_1}(\overline{v}-\underline{v}) + l_1 c] \\ &\quad -(1-\mu_1)^2(1-\mu_2)^k[\{2(1-\mu_2)^{l_2} + l_2\mu_2(1-\mu_2)^{l_2-1}\}(\overline{v}-\underline{v}) + l_2 c] - kc. \end{aligned} \tag{27}$$

First we observe that $k \leq n$ since the highest first-stage entry is achieved by setting the reserve price $r = \underline{v} + \varepsilon$. Define $l = n - k$ and rewrite expression for S^1 as follows:

$$\begin{aligned} S^1 &= 2\overline{v} - \{2\mu_1(1-\mu_1)(1-\mu_2)^k + (1-\mu_1)^2 k\mu_2(1-\mu_2)^{k-1}\}[(1-\mu_2)^{l}(\overline{v}-\underline{v}) + lc] \\ &\quad -(1-\mu_1)^2(1-\mu_2)^k[\{2(1-\mu_2)^{l} + l\mu_2(1-\mu_2)^{l-1}\}(\overline{v}-\underline{v}) + lc] - kc \\ &\quad -(1-\{2\mu_1(1-\mu_1)(1-\mu_2)^k + (1-\mu_1)^2[(1-\mu_2)^k + k\mu_2(1-\mu_2)^{k-1}]\})lc. \end{aligned} \tag{28}$$

Since l_1 and l_2 maximize expected surplus of the second stage when one and two units, respectively, are available, it follows that

$$\begin{aligned} (1-\mu_2)^{l_1}(\overline{v}-\underline{v}) + l_1 c &\leq (1-\mu_2)^{l}(\overline{v}-\underline{v}) + lc \\ \{2(1-\mu_2)^{l_2} + l_2\mu_2(1-\mu_2)^{l_2-1}\}(\overline{v}-\underline{v}) + l_2 c &\leq \{2(1-\mu_2)^{l} + l\mu_2(1-\mu_2)^{l-1}\}(\overline{v}-\underline{v}) + lc. \end{aligned}$$

Comparing (28) with (27) implies that $S^2 > S^1$. We summarize the result in the following lemma.

Lemma 9 *The expected surplus in a two-stage English auction is higher than in a standard English auction and an Anglo-Dutch auction.*

Observe that the previous argument easily extends to more than two units.

The revenues of the seller are the difference between net surplus and the profits of incumbents. We have already demonstrated that the surplus is higher in two-stage English auction than in the Anglo-Dutch auction. For the Anglo-Dutch auction to yield higher revenues it must be that the part of expected social surplus received by incumbents is smaller in the Anglo-Dutch auction than in the two-stage second price auction.

It can be shown, like in the case of one unit, that the revenues in a two-stage auction are maximized either when there is no entry in the first stage and we set the highest reserve price that induces incumbents with high valuations to bid in the first stage, or when the entry level in the first stage is (almost) socially efficient. (It will not be exactly socially efficient entry level because now profits of incumbents are not independent of entry k.) We provide partial results on revenue ranking in both auctions. Consider the case when $k = 0$ and

$$r = \overline{v} - [(1-\mu_1)\{(1-\mu_2)^{l_2} + l_2\mu_2(1-\mu_2)^{l_2-1}\} + \mu_1(1-\mu_2)^{l_1}](\overline{v} - \underline{v}).$$

Then the seller's revenues in the two-stage English auction are

$$\begin{aligned} R^2 \;=\; & 2\overline{v} - 2\mu_1(1-\mu_1)[(1-\mu_2)^{l_1}(\overline{v}-\underline{v}) + l_1 c] \\ & -(1-\mu_1)^2[\{2(1-\mu_2)^{l_2} + l_2\mu_2(1-\mu_2)^{l_2-1}\}(\overline{v}-\underline{v}) + l_2 c] \\ & -2\mu_1[\mu_1(1-\mu_2)^{l_1} + (1-\mu_1)\{(1-\mu_2)^{l_2} + l_2\mu_2(1-\mu_2)^{l_2-1}\}](\overline{v}-\underline{v}). \end{aligned} \tag{29}$$

It can be shown that the expected utility of incumbent in the Anglo-Dutch auction take the same expression as in (one-stage) English auction

$$\mu_1[(1-\mu_2)^n + (1-\mu_1)n\mu_2(1-\mu_2)^{n-1}](\overline{v}-\underline{v}). \tag{30}$$

Thus the expected revenue of the seller in the Anglo-Dutch auction can be written as

$$\begin{aligned} R^1 \;=\; & 2\overline{v} - 2\mu_1(1-\mu_1)[(1-\mu_2)^{n}(\overline{v}-\underline{v}) + nc] \\ & -(1-\mu_1)^2[\{2(1-\mu_2)^{n} + n\mu_2(1-\mu_2)^{n-1}\}(\overline{v}-\underline{v}) + nc] \\ & -2\mu_1[\mu_1(1-\mu_2)^{n} + (1-\mu_1)\{(1-\mu_2)^{n} + n\mu_2(1-\mu_2)^{n-1}\}](\overline{v}-\underline{v}) - \mu_1^2 nc \end{aligned} \tag{31}$$

It can be shown that the number of entrants in the Anglo-Dutch auction is given by

$$
\begin{aligned}
n = \max \quad & \{m|\mu_2 \left[\left\{ \mu_1^2(1-\mu_2)^{m-1} + \frac{1}{m} 2\mu_1(1-\mu_1)(1-\mu_2)^{m-1} \right. \right. \\
& \left. + \frac{2}{m(m+1)}(1-\mu_1)^2(1-\mu_2)^{m-1} \right\} \\
& \times \frac{\mu_1(1-\mu_2) + \frac{2}{m+1}(1-\mu_1)(1-\mu_2) + (1-\mu_1)\mu_2}{\mu_1(1-\mu_2) + \mu_1 m\mu_2 + \frac{2}{m+1}(1-\mu_1)(1-\mu_2) + (1-\mu_1)\mu_2} \\
& + \left\{ \frac{m-1}{m} 2\mu_1(1-\mu_1)(1-\mu_2)^{m-1} + 2\mu_1(1-\mu_1)(n-1)\mu_2(1-\mu_2)^{m-2} \right. \\
& \left. + \frac{4(m-1)}{m(m+1)}(1-\mu_1)^2(1-\mu_2)^{m-1} + \frac{2}{m}(1-\mu_1)^2(n-1)\mu_2(1-\mu_2)^{m-2} \right\} \\
& \times \frac{\frac{1}{m+1}(1-\mu_2)^2 + \mu_2(1-\mu_2)}{\frac{1}{m+1}(1-\mu_2)^2 + \mu_2(1-\mu_2) + \frac{m}{2}\mu_2^2} \\
& + \frac{(m-1)(m-2)}{(m+1)m}(1-\mu_1)^2(1-\mu_2)^{m-1} \\
& \left. + \frac{m-2}{m}(1-\mu_1)^2(m-1)\mu_2(1-\mu_2)^{m-2} \right] (\overline{v} - \underline{v}) \geq c\}.
\end{aligned} \tag{32}
$$

We want to know when the revenue of the seller is higher in the two-stage English auction (29), where entry is given by conditions (12) and (13), than in the Anglo-Dutch auction where entry is given by the condition (32). First, observe that when $\mu_1 = 0$, n is given by

$$\mu_2[(1-\mu_2)^{n+1} + (n+1)\mu_2(1-\mu_2)^n](\overline{v} - \underline{v}) = c$$

implying $n = l_2 - 2$. Differentiating the expression in square brackets of (32) with respect to μ_1 we obtain that the derivative is negative. The expression in square brackets of (32) is also declining with respect to n. Therefore we may conclude that higher probability μ_1 leads to lower entry n, and it is, at most, $l_2 - 2$. Assuming that $n = l_2 - 2$ for all μ_1 holds, when comparing (29) and (31), we obtain that $R^{2S} \geq R^{1S}$ if

$$
\begin{aligned}
& 2\mu_1(1-\mu_1)[(1-\mu_2)^{l_1}(\overline{v} - \underline{v}) + l_1 c] + 2\mu_1^2(1-\mu_2)^{l_1}(\overline{v} - \underline{v}) \\
\leq \; & 2\mu_1(1-\mu_1)[(1-\mu_2)^{l_2-2}(\overline{v} - \underline{v}) + (l_2-2)c] + 2\mu_1^2(1-\mu_2)^{l_2-2}(\overline{v} - \underline{v}) + \mu_1^2(l_2-2)c
\end{aligned}
$$

or

$$(1-\mu_2)^{l_1}(\overline{v}-\underline{v})+l_1c-\mu_1 l_1 c \leq (1-\mu_2)^{l_2-2}(\overline{v}-\underline{v})+(l_2-2)c-\frac{\mu_1}{2}(l_2-2)c.$$

The inequality will hold if $l_2 \leq 2(l_1+1)$, since $(1-\mu_2)^{l_1}(\overline{v}-\underline{v})+l_1c \leq (1-\mu_2)^{l_2-2}(\overline{v}-\underline{v})+(l_2-2)c$. (Because l_1 was chosen to minimize $(1-\mu_2)^{l_1}(\overline{v}-\underline{v})+l_1c$.)

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